ATMS542 Fluid Mechanics Review

Governing Equations

\[
\frac{D\vec{V}}{Dt} = -\frac{1}{\rho} \nabla H P - f \hat{k} \times \vec{V} \tag{1}
\]

\[
\frac{Dw}{Dt} = -\frac{1}{\rho} \frac{\partial P}{\partial z} - g \tag{2}
\]

\[
\frac{D\rho}{Dt} = -\rho \nabla \cdot \vec{U} \tag{3}
\]

\[
\frac{D\theta}{Dt} = 0 \tag{4}
\]

Notes

- Assumptions: neglected sphericity terms, friction, heating, and conduction.
- \( \vec{U} = (u, v, w) = \) velocity vector (m/s). \( \vec{V} \equiv (u, v) \).
- \( P = \) pressure (Pa); \( \rho = \) density (kg/m\(^3\)); \( T = \) temperature (K).
- \( P = \rho RT; R = 287 \text{ J/(K kg)}; \) technically \( T = T_v \), the virtual temperature.
- \( \theta = T \left( \frac{P_v}{P} \right)^{R/C_p} = \) potential temperature (K).
- \( g = \) effective gravity (m/s\(^2\)).
- \( f = 2\Omega \sin \phi = \) Coriolis parameter (s\(^{-1}\)); \( \phi = \) latitude.
- \( \frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{U} \cdot \nabla = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \)
  Lagrangian change = local change + advection.

Summary: 6 equations and 6 unknowns ( \( u, v, w, P, \rho, T \) )
Ertel potential-vorticity conservation law (incorporates all previous laws):

\[
\frac{D\Pi}{Dt} = 0. \quad \text{(adiabatic and frictionless)}
\]

where

\[
\Pi = \frac{\vec{\omega} + 2\vec{\Omega}}{\rho} \cdot \nabla \theta. \quad \text{(Ertel PV)}
\]

\[
\vec{\omega} = \nabla \times \vec{U} = (\xi, \eta, \zeta) \quad \text{(vorticity vector)}
\]

**Important Parameters and Dominant Force Balances**

Horizontal length scale: \((x, y) \sim L \sim 1000 \text{ km}\).

Vertical length scale: \(z \sim H \sim 10 \text{ km}\).

Aspect ratio: \(\delta = H/L \sim 0.01\).

Horizontal velocity scale: \((u, v) \sim U \sim 10 \text{ m/s}\). Since \(U \sim L/T\), this implies . . .

Time scale: \(t \sim T = L/U \sim 28 \text{ h}\).

Vertical velocity scale: \(w \sim H/T = \frac{HU}{L} = \delta U \sim 10 \text{ cm/s}\).

Ratio of horizontal accelerations to Coriolis effect (Rossby number):

\[
\frac{\rho D\vec{V}}{\rho f k \times \vec{V}} \sim \frac{U}{f L} \equiv \epsilon \sim 0.1
\]

Geostrophic balance in horizontal \((\epsilon \ll 1)\):

\[
0 \sim -\nabla_H P - \rho f \hat{k} \times \vec{V} \quad \Rightarrow \quad \vec{V} \approx \frac{1}{\rho f} \hat{k} \times \nabla_H P
\]

Hydrostatic balance in vertical \((\delta \ll 1)\):

\[
\frac{Dw}{Dt} \sim \epsilon \delta \ll 1 \quad \Rightarrow \quad \frac{\partial p}{\partial z} \approx -\rho g
\]

Note: (10) and (9) mean that accelerations are small relative to other terms.
**Constant-Density Limiting Fluids**

(1) **Shallow Water Equations**

Assumptions: $\rho = \text{constant}; \delta \ll 1; \text{adiabatic}.$

\[
\rho \frac{D\vec{V}}{Dt} = -\nabla H P - \rho f \hat{k} \times \vec{V}. \quad \frac{D\vec{V}}{Dt} = \frac{\partial}{\partial t} + \vec{V} \cdot \nabla H \tag{11}
\]

\[
\nabla H \cdot \vec{V} = -\frac{1}{h} \frac{Dh}{Dt} \quad (h = \text{fluid depth}) \tag{12}
\]

\[
\frac{\partial p}{\partial z} = -\rho g \tag{13}
\]

PV conservation (5)–(7) becomes

\[
\frac{D}{Dt} \left[ \zeta + f \frac{h}{h} \right] = 0. \tag{14}
\]

Note: because the fluid supports both *vorticity* and *divergence*, the velocity field may be represented by a nondivergent *streamfunction* plus an irrotational *velocity potential* (Helmholtz Theorem):

\[
\vec{u} = \nabla \times \vec{\Psi} + \nabla \chi \tag{15}
\]

\[
\nabla \times \vec{u} = \nabla \times \nabla \times \vec{\Psi} \quad \nabla \cdot \vec{u} = \nabla^2 \chi \tag{16}
\]
(2) Barotropic Vorticity Equation

Assumptions: Shallow water and $h = \text{constant}$.

PV conservation (14) becomes

$$\frac{D}{Dt}(\zeta + f) = 0. \quad (17)$$

Note: because this fluid supports only vorticity, and is strictly two dimensional [$\zeta = f(x, y)$], (15) allows the velocity field to be represented by a scalar nondivergent streamfunction:

$$\vec{V} = \nabla \times \hat{k} \psi = \hat{k} \times \nabla_H \psi \quad \zeta = \nabla_H \times \vec{V} = \nabla^2_H \psi \quad (18)$$

Using the $\beta$-plane approximation, $v \frac{\partial f}{\partial y} \approx v \beta$, and (17) becomes

$$\frac{D}{Dt} \zeta + v \beta = 0. \quad (19)$$

(19) has plane-wave solutions of the form

$$\psi(x, t) = Ae^{i(kx - \sigma t)} \quad (20)$$

provided

$$\sigma = -\frac{\beta}{k} \quad \text{(Rossby waves).} \quad (21)$$