NOTES AND CORRESPONDENCE

Comments on "The Roles of the Horizontal Component of the Earth's Angular Velocity in Nonhydrostatic Linear Models"

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1. Introduction

Kasahara (2003) recently investigated the linear normal modes arising in a series of problems designed to elucidate the influence of the horizontal component of the Coriolis force in Cartesian tangent-plane approximations to the earth's atmosphere or oceans. He suggests that when the horizontal component of the Coriolis force is included in the presence of horizontal rigid upper and and lower boundaries, "a distinct kind of wave oscillation emerges whose frequencies are very close to the inertial frequency," and refers to these waves as "boundary-induced inertial (BII) modes." In the case of a compressible stratified atmosphere he suggests dividing the normal modes into three types: "acoustic, inertio–gravity, and boundary-induced inertial modes." The same three pairs of eigenmodes were also recently identified in essentially the same type of tangent-plane analysis by Thuburn et al. (2002b), although no BII modes were found in a related numerical analysis of eigenmodes on the full sphere (Thuburn et al. 2002a). The purpose of this comment is to clarify the nature and physical significance of BII modes within the context of the tangent-plane approximation.

The distinction between gravity and acoustic waves is fundamental in that different physical processes (buoyancy and elastic restoring forces, respectively) are responsible for the propagation of each type of wave. No such distinction sets the BII mode apart from the inertio–gravity and acoustic modes; rather, as suggested by Kasahara, the mode appears in the set of normal modes due to the presence of the rigid upper and lower boundaries. Yet in contrast to a better known type of boundary-induced wave, the edge wave (in which the wave amplitude decays exponentially in the direction away from the boundary), the BII mode is simply the linear superposition of two waves that exist in an unbounded domain, and if properly excited, the BII "mode" could exist in the absence of the boundaries. In contrast, an edge wave such as a Kelvin or Lamb wave, cannot exist in the absence of a boundary because its amplitude would grow exponentially along one spatial direction. Moreover, because of its exponentially varying spatial structure, an edge wave cannot be exactly constructed from the linear superposition of a finite set of the sinusoidal wave solutions to the unbounded problem.

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2. Mathematics of the BII mode

The relation between the BII mode and more traditional inertial oscillations may be demonstrated by considering linear perturbations in a homogeneous rotating incompressible fluid, as in section 2 of Kasahara (2003). Assuming a vertical velocity distribution of the form

$$w = W(z)\exp[i(mx + ny - \sigma t)], \tag{1}$$

the vertical structure of $W(z)$ must satisfy

$$\frac{d^2W}{dz^2} + \frac{2if_Vf_Hn}{f_V^2 - \sigma^2}\frac{dW}{dz} + \left[\frac{\sigma^2(m^2 + n^2) - n^2f_H^2}{f_V^2 - \sigma}\right]W = 0. \tag{2}$$

Here, the notation follows Kasahara (2003), and, in particular, $f_V = 2\Omega\sin\phi$ and $f_H = 2\Omega\cos\phi$ are the vertical and horizontal components of the Coriolis force in the tangent-plane approximation at latitude ϕ.

In an unbounded domain, there are vertically propagating inertial wave solutions to (1) of the form

$$w = \exp[i(mx + ny + lz - \sigma t)], \tag{3}$$

provided σ satisfies the dispersion relation

$$\sigma^2 = \frac{(f_V l + f_H n)^2}{m^2 + n^2 + l^2} = \frac{4(\mathbf{\Omega} \cdot \mathbf{k})^2}{\mathbf{k} \cdot \mathbf{k}}, \quad (4)$$

where $\mathbf{k} = (m, n, l)$ is the wave vector, and $\mathbf{\Omega} = (0, f_H, f_V)$. Hereafter, we focus on the normal modes supported by a "horizontal channel" extending to infinity in the x and y directions, but bounded by rigid horizontal planes at $z = 0$ and $z = z_T$. Solutions of the form (3) do not satisfy the top and bottom boundary conditions in the horizontal channel ($w = 0$ at $z = 0$ and at $z = z_T$), but these boundary conditions can be accommodated by functions of the form

$$W(z) = w_0 \sin(kz) \exp(i\Gamma_2 z), \quad (5)$$

where

$$\Gamma_2 = -\frac{f_H f_V n}{f_V^2 - \sigma^2}. \quad (6)$$

Substitution of (1), (5), and (6) into (2) yields the following quadratic equation for σ^2:

$$(m^2 + n^2 + k^2)\sigma^4 - [(m^2 + n^2 + k^2)f_V^2 + n^2 f_H^2 + k^2 f_V^2]\sigma^2 + k^2 f_V^4 = 0. \quad (7)$$

Thus, as noted by Kasahara, for a given set of (m, n, k) (with $f_H n \neq 0$) there are two distinct values of $|\sigma|$, and four total roots to (7), associated with two distinct families of normal modes in the horizontal channel. In contrast, when there are no boundaries, or when the angular velocity vector is perpendicular to the boundaries ($f_H = 0$), there is only one value of $|\sigma|$ at which oscillations may occur, and that value of $|\sigma|$ is given by the dispersion relation (4).

In a horizontal channel with $f_H = 0$, by (6), $\Gamma_2 = 0$, so $l = k$, and the structure of each normal mode may be decomposed into the superposition of two vertically propagating waves with frequencies and wavenumbers (σ, m, n, l) for the wave with upward phase speed, and $(\sigma, m, n, -l)$ for the wave with downward phase speed. This familiar situation, in which normal modes are constructed as the superposition of vertically propagating waves, still holds when $f_H \neq 0$, but there are then two distinct ways to create suitable superpositions of vertically propagating waves, and as a consequence, two different values of $|\sigma|$ at which the normal modes may oscillate.

One simple way to appreciate that all the normal modes in the horizontal channel are the superposition of vertically propagating waves is to note that, according to (5), those normal modes are the sum of two waves of the form (3) with vertical wavenumbers $l = \Gamma_2 \pm k$. Indeed substituting $l = \Gamma_2 + k$ into the dispersion relation (4) and using (6), one may obtain the quartic polynomial relation (7). Thus, although there are two pairs of normal modes in the horizontal channel, and each pair oscillates at a different frequency, both pairs are superpositions of inertial wave solutions for an unbounded domain.

3. A conceptual approach to BII modes

The physical influence of the rigid boundaries on the solution, and the reason that two values for $|\sigma|$ arise when $f_H \neq 0$, may be better understood by writing (4) in the form

$$\sigma^2 = 4\Omega^2(\cos\theta)^2, \quad (8)$$

where θ is the clockwise angle between the angular velocity vector $\mathbf{\Omega}$ and the wave vector $\mathbf{k}$. To obtain a simple graphical diagram, consider two-dimensional waves in the y–z plane, so that the wave vector is reduced to (n, l). Without loss of generality, we focus on the case in which f_V, f_H, n, and σ are all positive. The normal modes in the horizontal channel are the superposition of pairs of vertically propagating waves whose wave vectors 1) have identical meridional components n, and 2) form angles with $\mathbf{\Omega}$ whose cosines are identical.

Let λ be the angle between $\mathbf{\Omega}$ and the vertical; note that $\lambda = \pi/2 - \phi$ is also the colatitude. The two families of normal modes consist of the "conventional" superposition of vertically propagating waves for which $|\theta| \geq \lambda$ and a "new" superposition of such waves, corresponding to Kasahara's BII modes, for which $|\theta| < \lambda$.

The structure of a conventional mode is illustrated in the left column of Fig. 1. For a given n there exist pairs of wave vectors (n, l_1) and (n, l_2) with identical values of σ forming angles α and $\pi - \alpha$ with respect to $\mathbf{\Omega}$. In order for the downward-pointing wave vector to have $n > 0$, one finds that $\lambda + (\pi - \alpha) < \pi$, or equivalently, $\lambda < \alpha$. To avoid double counting solutions, we also demand that $\alpha < \pi/2$. The specific pairs that can serve as normal modes to this problem are those for which l_1 and l_2 allow satisfaction of the rigid-plane boundary conditions $w = 0$ at $z = 0$ and $z = z_T$. Let the vertical velocity field be determined by the superposition

$$\sin(ny + l_1 z - \sigma t) - \sin(ny + l_2 z - \sigma t) = 2 \sin\left[\frac{(l_1 - l_2)z}{2}\right] \cos\left[ny + \frac{(l_1 + l_2)z}{2} - \sigma t\right].$$

With loss of generality, assume $l_1 > l_2$ as in Fig. 1; then the rigid boundary conditions will be satisfied provided that

$$l_1 - l_2 = \frac{2s\pi}{z_T}, \qquad s = 1, 2, \ldots. \quad (9)$$

Note that the y trace speeds (σ/n) of each vertically propagating wave and of the normal mode are all identical.

Conventional modes exist even when $\lambda = 0$, that is, when $f_H = 0$. In contrast, the new modes shown in right column of Fig. 1 are only present when $\lambda \neq 0$. In this

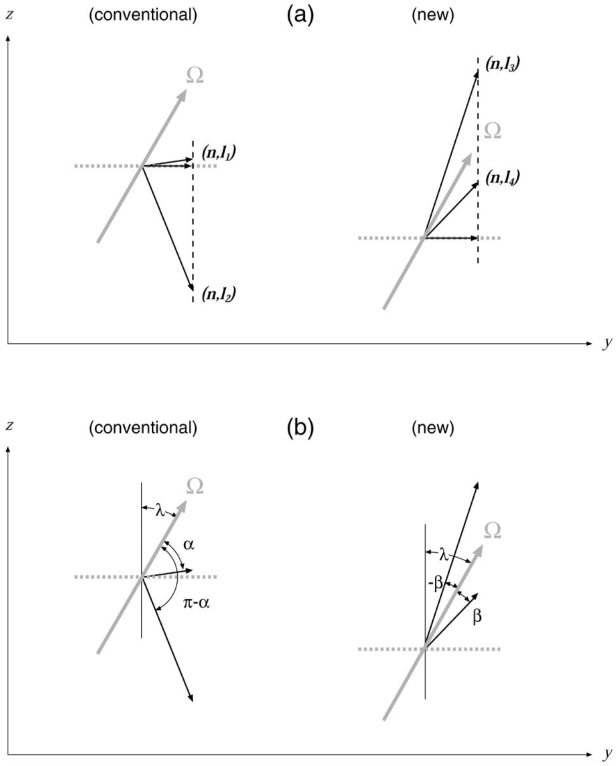

FIG. 1. (a), (b) Wave vectors $\mathbf{k} = (n, l_i)$ for pairs of vertically propagating waves that may be superimposed to form a normal mode in a horizontal channel rotating at angular velocity $\mathbf{\Omega}$ about an axis tilted off the vertical at angle λ. One rigid horizontal boundary is indicated by the dashed gray line. Panel (a) emphasizes that all $\mathbf{k}$ have the same meridional component n, and (b) illustrates that the cosine of the angle between $\mathbf{k}$ and $\mathbf{\Omega}$ is the same for all wave vectors.

case there are pairs of wave vectors (n, l_3) and (n, l_4) forming angles $\pm\beta$ with $\mathbf{\Omega}$, and there is the geometric constraint that $0 < \beta < \lambda$. As in the conventional case, all such pairs of vertically propagating waves have the same n and oscillate at the same σ. The actual pairs of vertically propagating waves that can serve as normal modes in the horizontal channel are those for which the vertical wavenumbers satisfy

$$l_3 - l_4 = \frac{2s\pi}{z_T}, \qquad s = 1, 2, \ldots .$$

What happens to these new modes in the limit that $f_H \to 0$? As $\lambda \to 0$ and the rotation vector becomes nearly vertical, $\beta \to 0$, $\sigma \to 2\Omega$, and the ratio of the horizontal to the vertical wavenumbers n/l approaches zero for both of the vertically propagating waves that superimpose to form the new-type modes. According to the incompressible continuity equation, the ratio of the vertical to the meridional velocity perturbations must also approach zero as $n/l \to 0$ in these two-dimensional waves. Thus, the $f_H \to 0$ limit of the new BII modes may be regarded as the family of inertial oscillations, which trivially satisfy (2) because $W = 0$.

In order for each pair of vertically propagating waves

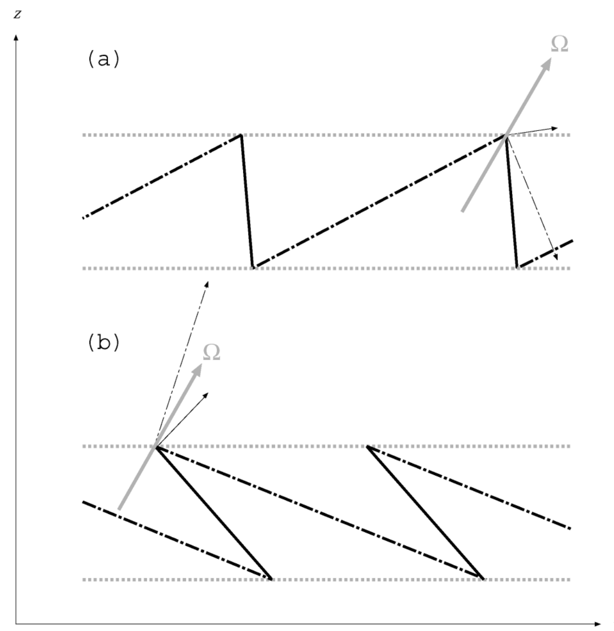

FIG. 2. (a), (b) Phase lines (heavy lines) and wave vectors (thin arrows) for the individual waves that superimpose to form a normal mode in a horizontal channel in which the angular velocity vector $\mathbf{\Omega}$ is not normal to the rigid boundaries. Phase lines and wave vectors for waves with upward group velocity are plotted using solid lines; dotted–dashed lines are used for the waves with downward group velocity. The gray short-dashed lines indicate the upper and lower boundaries of the horizontal channel. Only the mode shown in (a) can exist in the limit $f_H \to 0$.

to superimpose to form a *physically realizable* normal mode for a domain bounded by rigid upper and lower boundaries, one member of each pair should have upward group velocity while the group velocity for the other member is downward. Following the convention that n and l have arbitrary sign, but $\sigma > 0$ (to avoid redundant representations of the same wave), the following expression for the group velocities in the unbounded case may be obtained from (4):

$$\mathbf{c_g} = \left(\frac{\partial\sigma}{\partial n}, \frac{\partial\sigma}{\partial l}\right) = \mathrm{sgn}(\mathbf{k}\cdot\mathbf{\Omega})\left[\frac{f_H l - f_V n}{(n^2 + l^2)^{3/2}}\right](l, -n).$$

For the cases shown in Fig. 1, $n > 0$, and the sign of the vertical group velocity is given by

$$\mathrm{sgn}[(\mathbf{k}\cdot\mathbf{\Omega})\mathbf{i}\cdot(\mathbf{k}\times\mathbf{\Omega})], \qquad (10)$$

where $\mathbf{i}$ is the unit vector parallel to the x axis. According to (10), the group velocities of the waves shown in Fig. 1 with vertical wavenumbers l_1 and l_4 are upward, while those of the other two are downward. Thus, the physical realizability condition is satisfied for both the conventional and the new modes.

The structure of the two different types of normal modes is further illustrated in Fig. 2, which shows representative phase lines for the vertically propagating inertial waves considered in Fig. 1. The plane rigid boundaries of the horizontal channel are indicated by the short gray dashed lines, and the angular velocity and wave vectors are plotted in the *y*–*z* plane just as in Fig. 1. Lines of constant phase, which are perpendicular to the wave vectors, are plotted as heavy lines with a solid or dotted–dashed pattern matching that used to display the corresponding wave vector. The phase lines shown may be considered extrema that are 180° out of phase, so that the individual waves superimpose to form nodal surfaces at each boundary. The waves with upward group velocity are represented by solid lines, those with downward group velocity are shown with dotted–dashed lines.

4. BII modes in stratified flow

The graphical analysis in Fig. 1 can easily be generalized to include density-stratified fluids with buoyancy frequency N. For simplicity, the focus is limited to waves in the meridional *y*–*z* plane. Let γ be the clockwise angle from the vertical to the wave vector $\mathbf{k}$. As before, define λ to be the clockwise angle between the vertical and the angular rotation vector $\mathbf{\Omega}$, and let $\theta = \gamma - \lambda$ be the clockwise angle from $\mathbf{\Omega}$ to $\mathbf{k}$. The dispersion relation for vertically propagating plane waves in a rotating Boussinesq fluid satisfies (3.21) of Kasahara (2003), or equivalently,

$$\sigma^2(\gamma) = N^2 \sin^2(\gamma) + 4\Omega^2 \cos^2(\gamma - \lambda). \quad (11)$$

Without loss of generality we consider only positive meridional wavenumbers $n > 0$, for which any choice of vertical wavenumber l will give $0 < \gamma < \pi$. Normal modes for a horizontal channel bounded by rigid planes at $z = 0$ and $z = z_T$ are obtained by superimposing pairs of vertically propagating waves with the same σ and n, but with l's that differ by an integer multiple of $2\pi/z_T$.

Writing (11) in the equivalent form

$$\sigma^2 = \frac{1}{2}N^2(1 - \cos 2\gamma) + 2\Omega^2(1 + \cos 2\gamma \cos 2\lambda + \sin 2\gamma \sin 2\lambda)$$

demonstrates that σ^2 is a π-periodic sinusoidal function of γ. In the interval $0 < \gamma < \pi$, there will be a single maximum of σ^2 at some angle $\gamma_{\max}$ and a single minimum value at some other angle $\gamma_{\min}$. These extrema occur when

$$0 = \frac{d\sigma^2}{d\gamma} = N^2 \sin 2\gamma - 4\Omega^2 \times (\sin 2\gamma \cos 2\lambda - \cos 2\gamma \sin 2\lambda), \quad \text{or}$$

$$\tan 2\gamma = \frac{-4\Omega^2 \sin 2\lambda}{N^2 - 4\Omega^2 \cos 2\lambda}. \quad (12)$$

Between $\gamma_{\min}$ and $\gamma_{\max}$, there will be a unique angle γ_c at which $\sigma^2(\gamma_c) = \sigma^2(0) = f_V^2$. The only exceptions, which we will not consider further, are if $\mathbf{\Omega}$ is vertical (the case $\lambda = 0$) or horizontal ($\lambda = \pi/2$), in which case

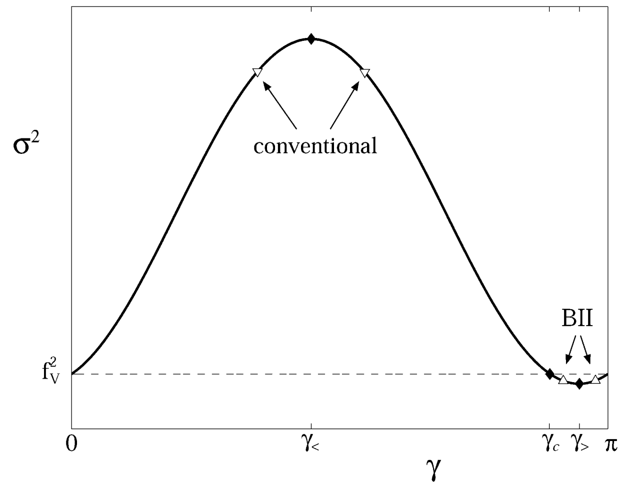

FIG. 3. Squared frequency of plane waves making up the normal modes of a stratified Boussinesq fluid rotating at angular velocity $\mathbf{\Omega}$ about an axis tilted at angle λ from the vertical in a horizontal channel plotted as a function of the clockwise angle between the wave vector and the vertical when $N^2/(4\Omega^2) = 3$ and λ is 30°.

one extremum is on the boundary of the interval. Let $\gamma_<$ be the smaller of $\gamma_{\min}$ and $\gamma_{\max}$, and $\gamma_>$ be the larger of these two angles. The points $\sigma^2(\gamma_<)$, $\sigma^2(\gamma_c)$, and $\sigma^2(\gamma_>)$, are plotted in Fig. 3 for the case in which $\lambda = \pi/6$ and $N^2/(4\Omega^2) = 3$.

Now consider how a normal mode in the horizontal channel may be constructed from pairs of vertically propagating waves with a given meridional wavenumber n. Let the two plane waves that make up a candidate mode make clockwise angles $0 < \gamma_1 < \gamma_2 < \pi$ to the vertical. Since the two plane waves must have equal frequencies, there are always two possible types of bounded modes analogous to those shown in Fig. 1. The first type has $0 < \gamma_1 < \gamma_< < \gamma_2 < \gamma_c$, as illustrated by the downward-pointing triangles in Fig. 3. The second type has $\gamma_c < \gamma_1 < \gamma_> < \gamma_2 < \pi$, and is represented by the upward-pointing triangles in Fig. 3. For either type of mode, as γ_1 ranges over all permissible values, the wavenumber differences $l_1 - l_2 = n[\cot(\gamma_1) - \cot(\gamma_2)]$ will vary from 0 (when both plane waves would have the extremal frequency) to ∞ (when one plane wave has nearly vertical wavenumber and the other has angle γ_c). Thus, for any n and nonzero integer s, one can always find exactly one pair of angles γ_1 and γ_2 within each mode class (or equivalently, one pair of vertical wavenumbers l_1 and l_2) that satisfies the boundary condition (9).

In the geophysically relevant case $N \gg 2\Omega$, σ^2 will have a maximum close to N^2, and the corresponding normal mode is a rotationally perturbed buoyancy oscillation with a wave vector pointing slightly above the horizontal ($\gamma_{\max}$ slightly less than $\pi/2$). The minimum frequency, corresponding to a nearly pure inertial wave perturbed by stratification, will occur for $\gamma_{\min}$ slightly less than π; in particular, (12) implies $\pi - \gamma_{\min} =$

$O(\Omega^2/N^2)$. The crossover angle γ_c between conventional and BII modes is less than $\gamma_{\min}$ by a similarly small amount. Thus, the new BII modes all consist of plane waves with wave vectors that point nearly straight down, and as pointed out by Kasahara and others, these modes resemble inertial oscillations.

5. Conclusions

The preceding analysis generalizes to other simple types of waves in the presence of ''tilted'' rigid upper and lower boundaries. In particular, vertically propagating internal gravity waves in a nonrotating ($f_H = f_V = 0$) Boussinesq fluid satisfy the dispersion relation

$$\sigma^2 = N^2 \sin^2\gamma,$$

where γ is again the angle between the vertical coordinate and the wave vector. In the familiar case where the rigid boundaries are horizontal, for each set of wavenumbers (m, n, k) there are only two normal modes, both oscillating at the same $|\sigma|$. If however, the boundary is tilted with respect to the vertical, so that the gravitational restoring force is no longer normal to the boundaries, two sets of normal modes may be constructed for each (m, n, k) in precisely the same way illustrated in Fig. 1, except that the vector labeled $\mathbf{\Omega}$ should be relabeled as $\mathbf{N}$.

Returning to the interpretation of the BII modes in Kasahara (2003), we would assert that, from a dynamical standpoint, these modes are essentially identical to the usual inertial modes appearing in the $f_H = 0$ case. The BII modes are simply a second way that vertically propagating inertial waves can be superimposed to satisfy the boundary conditions when the earth's angular velocity vector is not normal to the boundaries. In Boussinesq stratified flow both the ''inertio–gravity'' and ''boundary-induced inertial'' normal modes are the superposition of two vertically propagating inertio–gravity waves. The frequencies and wavenumbers for these two types of modes can be very different, but in comparison with the dynamical difference between acoustic and gravity waves, the dynamics that underlie inertio–gravity and BII modes are essentially the same. Finally, the influence of boundaries on BII modes is less profound than that for ''edge waves'' in which the presence of the boundary introduces waves with spatial structures not permitted in the unbounded domain.

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REFERENCES

Kasahara, A., 2003: The roles of the horizontal component of the earth's angular velocity in nonhydrostatic linear models. *J. Atmos. Sci.,* **60,** 1085–1095.

Thuburn, J., N. Wood, and A. Staniforth, 2002a: Normal modes of deep atmospheres: I. Spherical geometry. *Quart. J. Roy. Meteor. Soc.,* **128,** 1771–1792.

——, ——, and ——, 2002b: Normal modes of deep atmospheres: II. f–F-plane geometry. *Quart. J. Roy. Meteor. Soc.,* **128,** 1793–1806.