A Two-Box Model of Cloud-Weighted Sea-Surface Temperature:
The Semi-Automatic Negative Correlation with Mean Cloud Fraction

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The system of interest consists of two equal areas with SSTs of \( T_c \) and \( T_w \) and cloud area fractions of \( C_c \) and \( C_w \). In this case the cloud-weighted SST (CWT) and average cloud cover, \( A \), are given by:

\[
\text{CWT} = \frac{C_w T_w + C_c T_c}{2A} \quad \text{and} \quad A = \frac{1}{2} (C_w + C_c)
\]  

(1)

We will assume that the SST remains fixed with time, so that the variations in CWT arise solely from variations in cloud coverage. Defining \( \Delta T = T_w - T_c \), and noting that

\[
C_w = 2A - C_c
\]

we can write:

\[
\text{CWT} = T_w - \frac{C_c}{C_c + C_w} \Delta T = T_c + \frac{C_w}{C_c + C_w} \Delta T
\]  

(2)

Intuitively, it seems that if the variance of the cloud cover in the cold region is greater than the variance in the warm region, then the ratio \( \frac{C_c}{C_c + C_w} \) will be positively correlated with \( A \) and the ratio \( \frac{C_w}{C_c + C_w} \) will be negatively correlated with \( A \), so that \( \text{CWT} \) will perforce be negatively correlated with \( A \). The following mathematical analysis of this intuition was provided to us by our colleague Christopher S. Bretherton.
The correlation between the mean cloud cover $A$ and the factor $F = \frac{C_c}{C_c + C_w}$ is proportional to the covariance of $F$ with $2A$, $\text{Cov}\{F, 2A\}$. Assume that the cloudiness values in the two regions are random time series with known means and standard deviations.

$$C_c = \bar{C}_c + \sigma_c n_c$$
$$C_w = \bar{C}_w + \sigma_w n_w$$

where $n_c$ and $n_w$ are random timeseries with zero mean and unit variance. Further define $r = \text{Cov}\{n_c, n_w\}$, the correlation between the random time series that make up the time-varying part of the cloudiness timeseries for the two regions. It will also be necessary to assume that

$$\frac{\sigma_c}{C_c} \ll 1 \text{ and } \frac{\sigma_w}{C_c + C_w} \ll 1$$

Then,

$$2A = C_c + C_w = \bar{C}_c + \bar{C}_w + \sigma_c n_c + \sigma_w n_w$$

and

$$F = \frac{C_c}{C_c + C_w} = \frac{\bar{C}_c}{\bar{C}_c + \bar{C}_w} \left( 1 + \frac{\sigma_c n_c}{\bar{C}_c} \right) \left( 1 + \frac{\sigma_w n_w}{\bar{C}_w} \right)$$

so that,

$$\text{Cov}\{F, 2A\} = \frac{\bar{C}_c}{\bar{C}_c + \bar{C}_w} \text{Cov}\left\{ \frac{\sigma_c n_c}{\bar{C}_c} - \frac{\sigma_c n_c + \sigma_w n_w}{\bar{C}_c + \bar{C}_w}, \sigma_c n_c + \sigma_w n_w \right\}$$

If $r = 0$, then $\text{Cov}\{F, 2A\} > 0$ under the condition that,
\[ \frac{\sigma_c^2}{\sigma_w^2} > \frac{\overline{C_c}}{C_w} \] (9)

The same condition (9) will guarantee that,

\[ \text{Cov}\left\{ \frac{C_w}{C_c + C_w}, C_c + C_w \right\} < 0 \] (10)

These derivations suggest that a negative correlation between cloud-weighted SST and cloud amount should be expected if the cloud amount in the cold region has relatively low mean cloudiness and/or high variability compared to the warm area.