6.4 Singular Spectrum Analysis

One can use eof analysis to look for structure in a time series by doing an eigenanalysis of the lagged covariance matrix. It has been argued that this is useful in nonlinear systems analysis because you don't have to choose the structure functions a priori, but you let the data themselves choose the temporal structures. See Ghil et al.(2002) for a review paper.

Suppose that we have a time series x of length N. Suppose that we choose some maximum lag M and compute the $M \times M$ lag autocovariance matrix, .

$$C_{ij} = \frac{1}{N-M} \sum_{k=1}^{N-M} x(k+i)\,x(k+j) \tag{6.109}$$

In the notation we've been using, the easiest way to compute this lagged correlation matrix is to take the data vector x and make it into an N' x M augmented matrix, where N' = N – M +1. This augmented matrix repeats the time series x in each row, but lagged by one time step.

$$\mathbf{A} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & \cdots & x_{N-M+1} \\ x_2 & x_3 & x_4 & x_5 & x_6 & \cdots & x_{N-M+2} \\ x_3 & x_4 & x_5 & x_6 & x_7 & & x_{N-M+3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_M & x_{M+1} & x_{M+2} & x_{M+3} & x_{M+4} & \cdots & x_N \end{bmatrix} \tag{6.110}$$

Then the lagged covariance matrix we want is simply,

$$\mathbf{C} = \mathbf{A}\mathbf{A}^T / (N-M+1) \tag{6.111}$$

If we do an eigenanalysis of this covariance matrix, we will obtain temporal structures that explain the maximum possible amount of the temporal autocovariance on an interval of measure M.

$$\mathbf{U}\Lambda\mathbf{U}^T = \mathbf{C} \tag{6.112}$$

The eigenvalues will be the amount of covariance in time explained by each eigenvector. You can then reconstruct the time series at any time, by expanding the data in the basis set of eigenfunctions.

$$\mathbf{Z} = \mathbf{U}^T\mathbf{A} \qquad \mathbf{A} = \mathbf{U}\mathbf{Z} \tag{6.113}$$

It is very easy to write an m-file to do this kind of 'singular spectral analysis' in Matlab. It is interesting to see what such a program does with time series of various types, and how the shapes you get are sensitive to the window length M. Let's start by looking at a red noise time series, with a one-lag autocorrelation of 0.5. Let's set N=4000 and M=50, to start with. The eigenvalue spectrum and the first three eigenvectors are shown below.

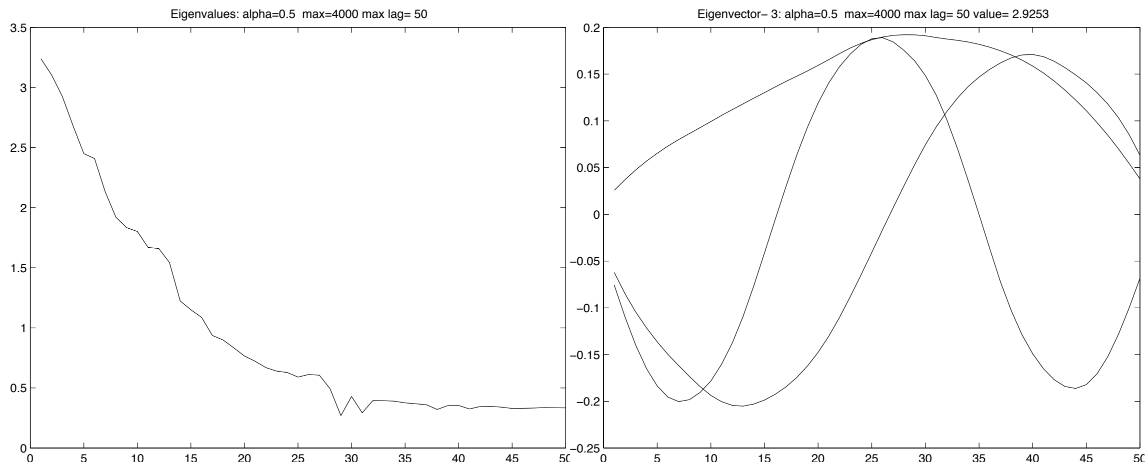


Fig. 6.27 *SSA analysis of Red Noise with an autocorrelation of 0.5, using a window of length 50 time steps. The eigenvalue spectrum is on the left and the first three eigenvectors on the right. The first EOF is the smoothest.*

If we keep everything the same, except we make the data window 70 instead of 50, the eigenvalue spectrum remains similar, but the eigenvectors change to fit the domain.

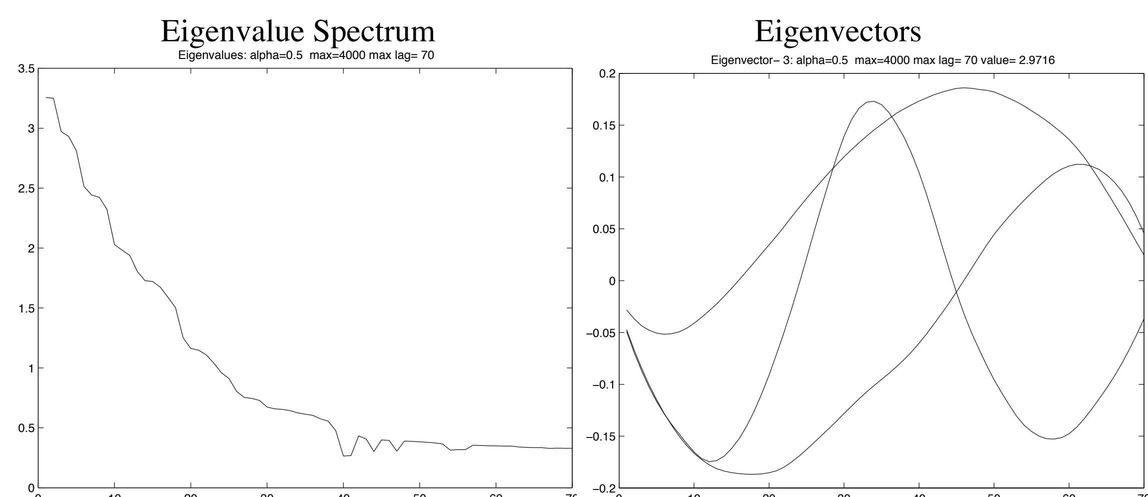


Fig. 6.28 *Same as Fig. 7.27, except that the data window (embedding dimension) is taken to be 70 instead of 50.*

It has been argued that SSA might be better than Fourier spectral analysis because the structures are not determined a priori, but rather are determined from the structure in the data themselves. I doubt if this is true in general, for the following reasons. SSA is still a linear RMS fitting method. Fourier analysis is optimal in an RMS sense. Therefore, the functions we are likely to get from SSA will have shapes similar to the Fourier modes on the interval M that we choose. The eigenvalues will order these empirical Fourier modes in such a way that the dominant periods will appear first. This is already somewhat apparent from the analysis of red noise above. The structure functions look like Fourier modes. Let's now suppose that the input time series is a saw tooth wave with a period of 20 time units, as pictured below.

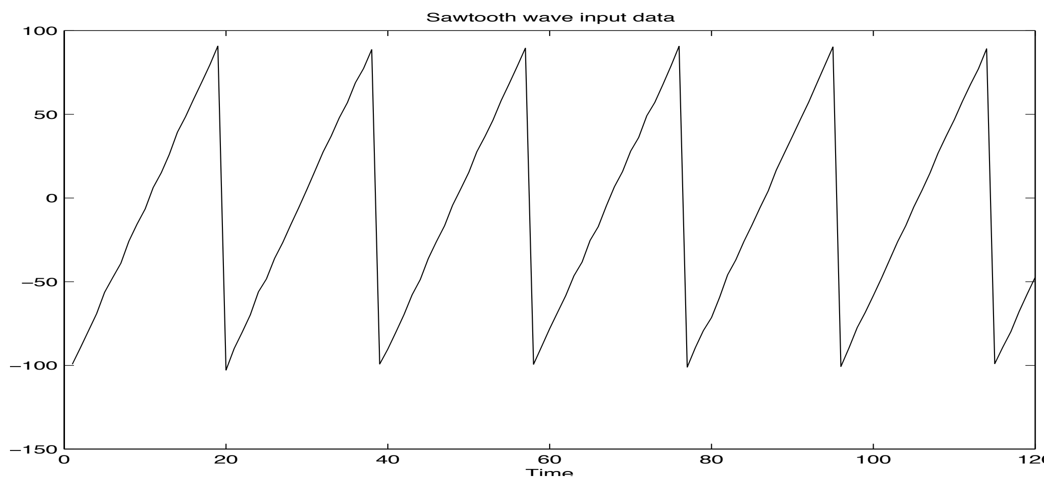


Fig. 6.29 *A sawtooth wave with a period of 20 time units.*

Let's take 4,000 units of this and subject it to SSA with a window length of M=80. This should give the SSA method a good chance to come up with a sawtooth structure function, if it can, since we have chosen an embedding dimension that is an integral number of sawtooth wave period.. As seen below, in fact it doesn't. It chooses to use Fourier modes with the appropriate length scales. The first three temporal structures are indistinguishable from sines and cosines with the appropriate scales to explain the input data. We could have gotten the same information from Fourier spectral analysis. An advantage of Fourier spectral analysis is that the techniques for determining the statistical significance of the results are better developed.

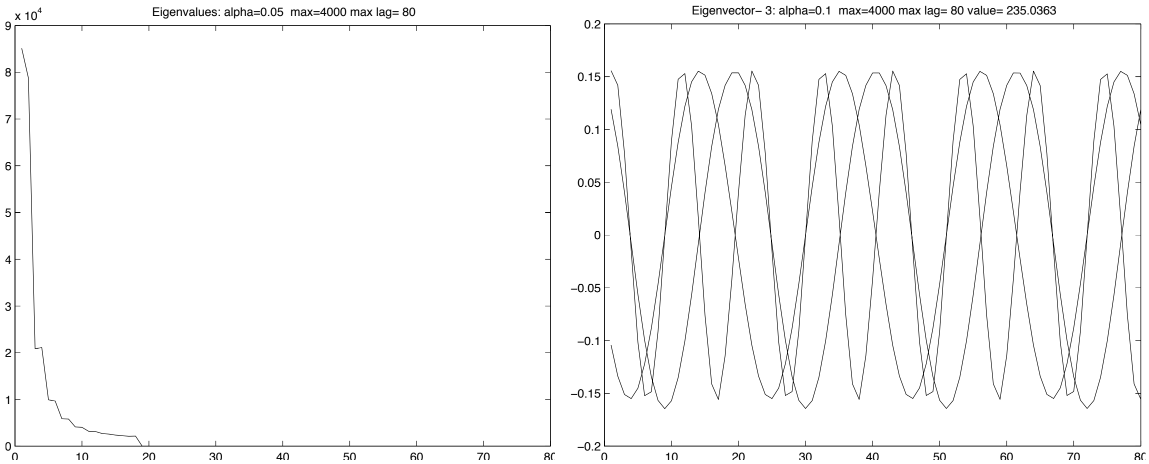


Fig. 6.30 *SSA analysis of a sawtooth wave with a period of 20 time units, using a window of length 80 time steps. The eigenvalue spectrum is on the left and the first three eigenvectors on the right.*

6.4b SSA: Dynamical Systems Approach

Broomhead and King(1986) suggested using a very small embedding dimension to look for structure in time series data from dynamical systems. Much of the emphasis was placed on looking at the phase diagrams, which are the trajectories of the PCs of the first few modes. This is designed to seek not periodicities, but rather repeated trajectories in phase space. We can illustrate this with the sawtooth time series with added noise. Noise makes things more problematic. In geophysics we are almost always dealing with high noise systems, if they are also complex systems. As our test time series we use a sawtooth time series with various levels of noise present. We consider cases in which the signal is ten times, three times and two times the noise level. A portion of each of these time series is shown below.

Sawtooth is 10 times noise amplitude

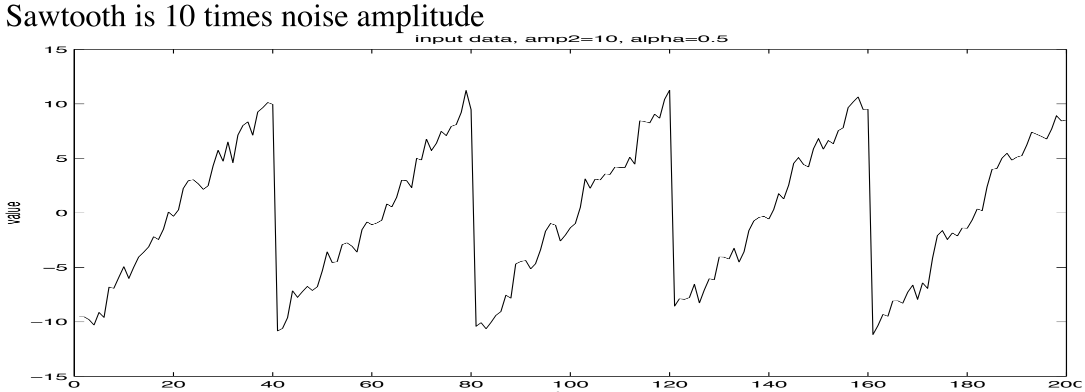


Fig. 6.31 *A sawtooth wave with a period of 20 time units, with added noise that is 1/10 as big as the signal.*

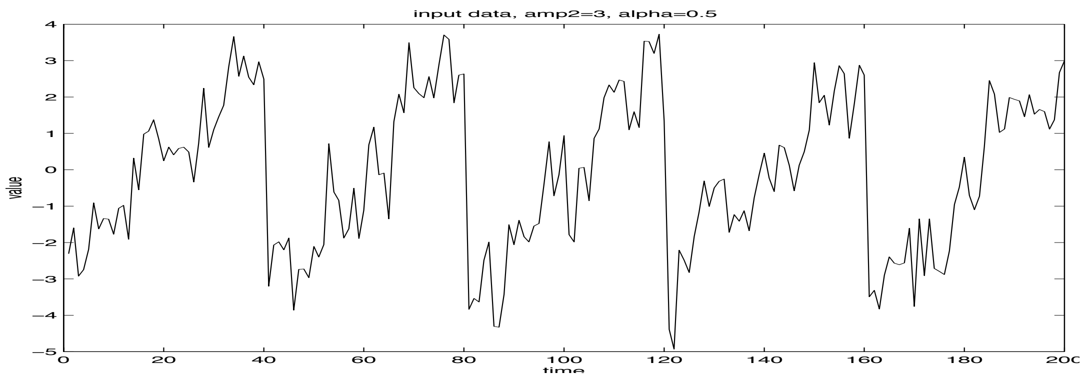


Fig. 6.32 *A sawtooth wave with a period of 20 time units, with added noise that is 1/3 as big as the signal.*

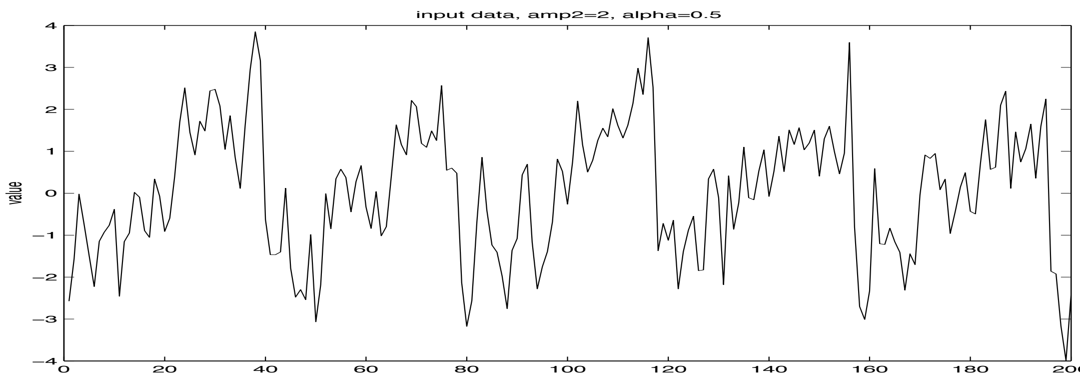
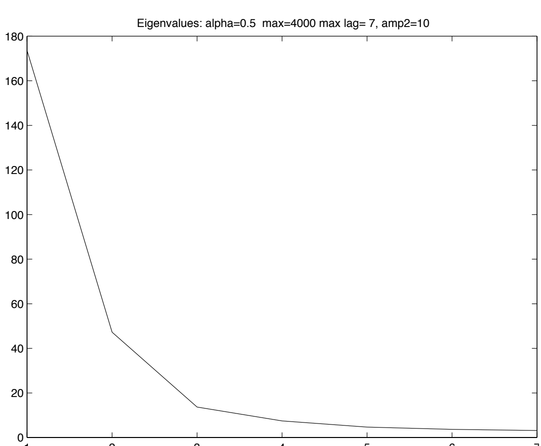
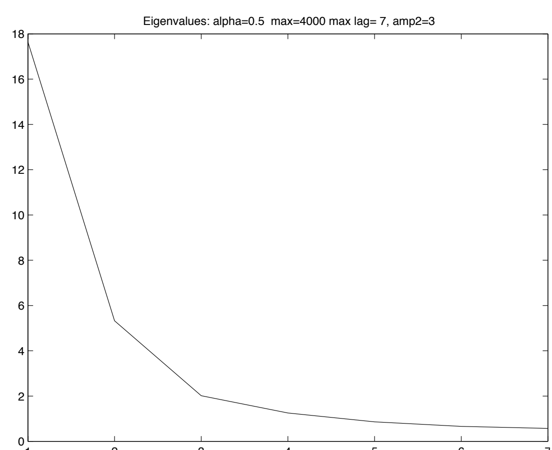


Fig. 6.33 *A sawtooth wave with a period of 20 time units, with added noise that is 1/2 as big as the signal.*

We compute eigenvalue spectra for embedding dimension of 7 for the three amplitude levels of 10, 3, and 2 times the noise level are shown below. The shapes are similar, but the magnitude of the eigenvalues are different because of the amplitudes we put in.





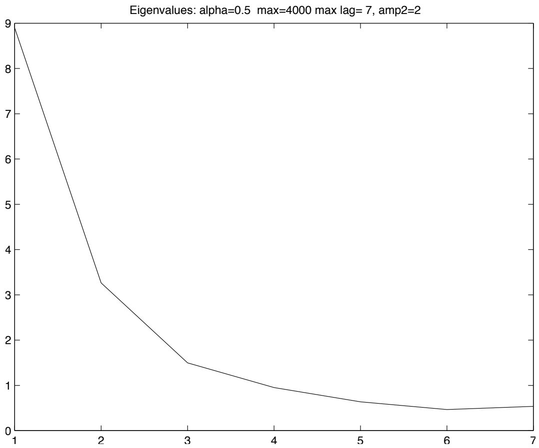
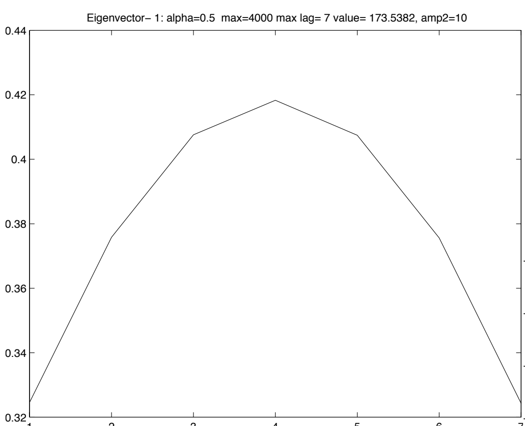
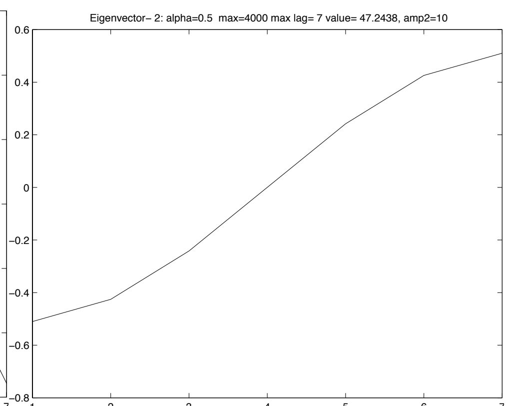


Fig. 6.34 *Eigenvalue spectra for SSA of the sawtooth wave with three levels of noise. Note how similar they look in terms of basic shape.*

Below are the first three eigenvectors for the SSA analysis of embedding dimension 7 for the case with the sawtooth amplitude 10 times the noise amplitude. The eigenvectors are very simple, almost harmonic, structures. The first is nearly a constant. The second is a trendline. The third is like a cosine wave. These do not change much as the noise increases relative to the signal. They are picking out the amplitude, trend and curvature of the time series on an interval of 7.





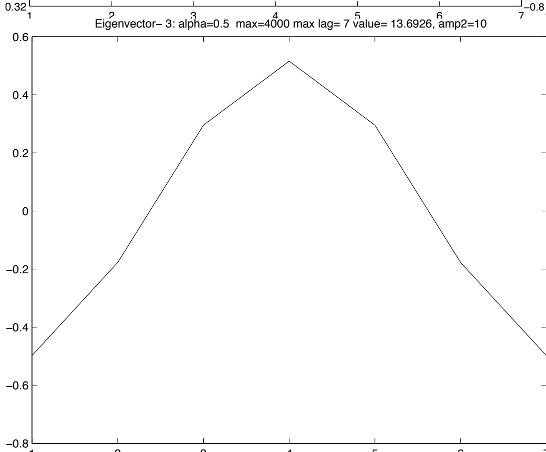


Fig. 6.35 *Eigenvectors for SSA of the sawtooth wave with three levels of noise.*

Next we show phase plots of the principal components of the first two structure functions for the cases with amplitude 10, 3 and 2. The PCs are related in time as they trace out the sawtooth. So we get well-defined trajectories in phase space. Note that the phase trajectories are less well defined as the noise level gets greater relative to the sawtooth signal.

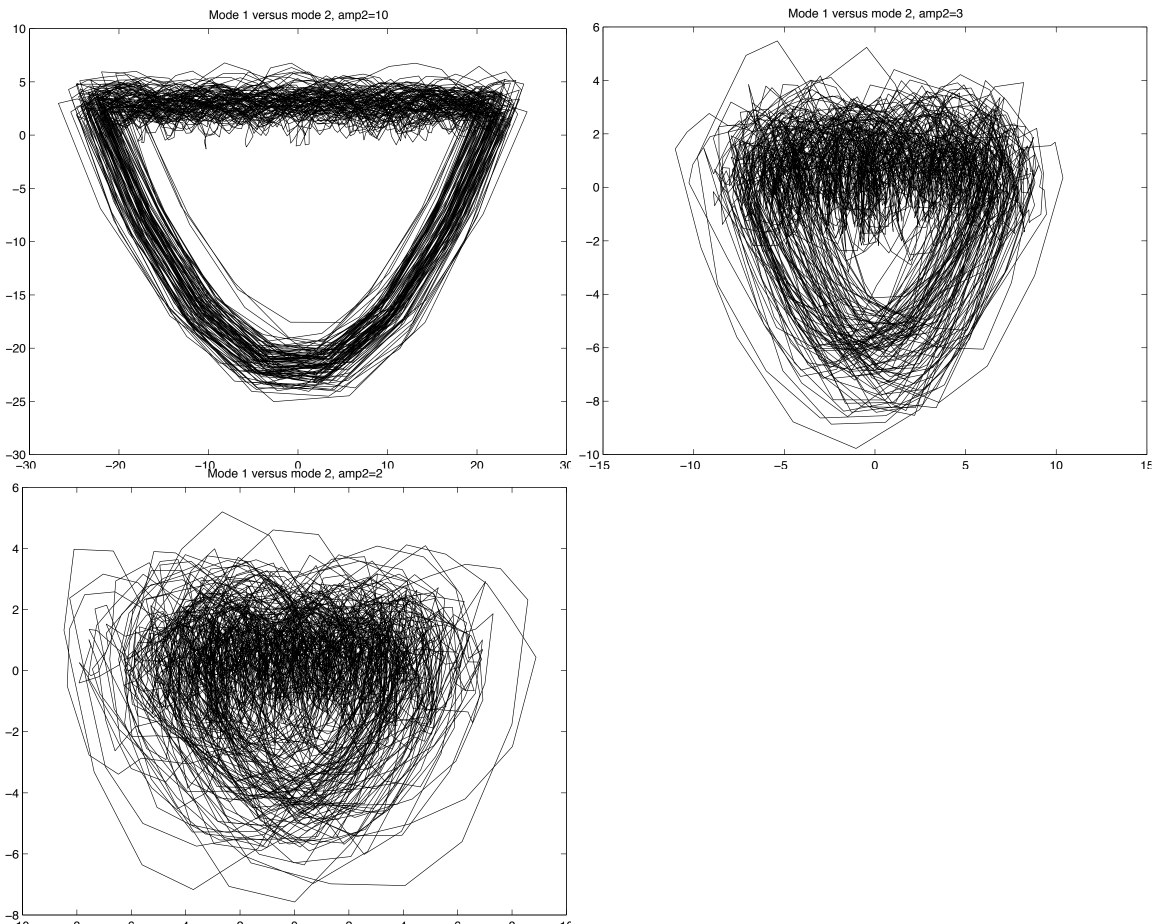


Fig. 6.36 *Phase plots where the PC's of the first two eigenvectors of the SSA as a function of time, are plotted versus each other.*

Below we show the same comparison for the phase plots of PC 1 and 3, for the cases of amplitude 10 and 3, and we see similar things.

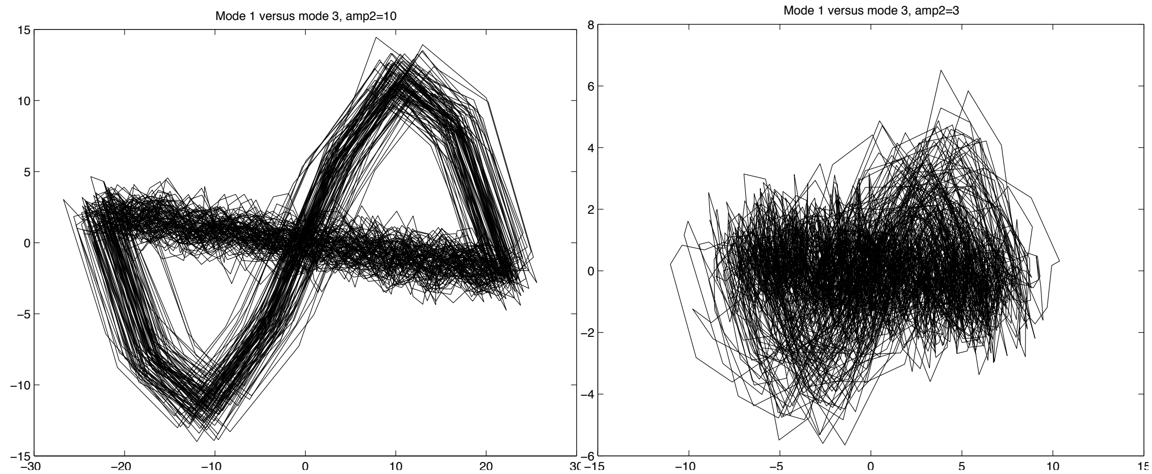


Fig. 6.37 *Same as Fig. 6.37, except for the first and third eigenvectors of the SSA.*

So perhaps if you have a dynamical system with low noise, but some organized chaotic behavior, you can detect the attractor trajectories using this kind of small embedding dimension singular spectral analysis. Broomhead and King(1968) show that this works for the Lorenz model.

6.5 Maximum Entropy Spectral Analysis:

Maximum entropy spectral analysis is a technique that can be used when you have a short period of record, but you need more spectral resolution that you can get by doing traditional Fourier Spectral analysis on the available data (Press, et al. 1992, page 565). It provides this extra spectral resolution by extending the autocovariance matrix in a way that adds the least information to the covariance matrix (maximum entropy). It will tend to strongly localize spectral peaks, so you can determine their location very precisely. The problem is that it has adjustable parameters that can be used to get a spectrum of arbitrary sharpness. The tools for assigning statistical significance to the results of such an analysis are not yet developed. It is best used in conjunction with traditional Fourier spectral analysis, after you have established the significance of periodicities in the time series of interest.

6.6 Multi-Taper Method

In the Welch overlapped segment averaging (WOSA, Welch, 1967) method that we described in section 6.2.5, a Hamming window was applied to individual segments that over overlapped by 50% to give a spectrum with minimal leakage, a high number of degrees of freedom and which uses the data effectively except near the beginning and end of the time series. The Hamming window is tapered, which gives the spectrum its desired frequency response. As the name implies, the multi-taper method (Thomson, 1982, 1990a) uses a set of different data tapers to try to provide an optimal estimate of the spectrum of the time series. A nice description is given in Percival and Walden(1993). The idea is to choose a set of tapers that provide optimal resolution, minimum leakage and are orthogonal. One comes up with a set of such tapers that are a set of harmonics, and then one uses the average of spectra computed using a number of these tapers, rather than just choosing one based on minimizing the negative side lobes.

6.7 Higher-Order Spectral Analysis

For nonlinear systems, the linear reasoning underlying the analysis of variance may not capture the relevant dynamics. For this reason people are beginning to experiment with higher-order polyspectra (e.g. Nikias and Petropulu, 1993). These higher order spectra will be zero if the variable has a Gaussian distribution. The simplest thing to consider is the bispectrum, which is obtained as the Fourier Transform of the bicovariance function. The bispectrum is a two dimensional contour plot, since frequency pairs are involved instead of the power at a single frequency.

References on Spectral Analysis

Allen, M.R., and Smith, L.A., 1994: Investigating the origins and signficance of low-frequency modes of climate variability, Geophys. Res. Letters, 21, 883-886.

Allen, M.R., Read, P.L., and Smith, L.A., 1992a: Temperature time-series?, Nature, 355, 686.

Allen, M.R., Read, P.L., and Smith, L.A., 1992b: Temperature oscillations, Nature, 359, 679.

Allen, M.R., Read, P.L., and Smith, L.A., 1992c: Investigating interdecadal temperature variations using singular spectrum analysis, Ann. Geophysicae, 10 suppl. 2, C253.

Amos, D. E. and L. H. Koopmans, 1963: Tables of the Distribution of the coefficient of coherence for stationary bivariate Gaussian ProcessesSCR 483, 328 pp.

Bath, Marcus, 1974: *Spectral Analysis in Geophysics*. Elsevier, Amsterdam, 563 pp.

Bingham, C., M. D. Godfrey and J. W. Tukey, 1967: Modern techniques of power spectrum estimation. IEEE Trans. Audio Electroacoustics, AU-15, 56-66.

Blackman, R. B., and J. W. Tukey, 1959: The Measurement of Power Spectra. Dover, 190 pp.

Blackman, R.B., and Tukey, J,W., 1958: The measurement of power spectra from the point of view of communication engineering. Dover Publications, 190 pp.

Box, G.E.P., and Jenkins, G.M., 1976: Time series analysis: forecasting and control. Holden-Day, Oakland, Cal., 575 pp.

Broomhead, D.S., and King, G., 1986: Extracting qualitative dynamics from experimental data, Physica D, 20, 217-236.

Burg, J.P., 1978: A new technique for time series data, in Childers, 1978, IEEE Press.

Childers, D.G., 1978: *Modern Spectrum Analysis*, IEEE Press (reprint series), New York, 334 pp.

Childers, D.G., ed., 1978: Modern Spectrum Analysis. IEEE Press, 334 pp.

Colebrook, J.M., 1978: Continuous plankton records - zooplankton and environment, northeast Atlantic and North-Sea, 1948-1975, Oceanol. Acta, 1, 9-23.

Elsner A.A., and Tsonis, J.B., 1991, Do bidecadal oscillations exist in the global temperature record?, Nature, 353, 551-553.

Fraedrich, K., 1986: Estimating the dimension of weather and climate attractors, *J. Atm. Sci.*, **43**, 419-432.

Ghil, M., and Vautard, R., 1991: Interdecadal oscillations and the warming trend in global temperature time series, Nature, 350, 324-327.

Ghil, M. et. al. 2002: Advanced Spectral Methods for Climatic Time Series. *Rev. Geophys.*, ***40***, DOI 10.1029/2001RG000092.

Gilman, D., P. Fuglister, and J.M. Mitchell, 1963: On the power spectrum of red noise. J. Atmos. Sci., 20, 182-184.

Hayashi, Y., 1971, A generalized method of resolving disturbances into progressive and retrogressive waves by space Fourier and time cross-spectral analysis. *J. Meteor. Soc. Japan*, **49**, 125-128.

Hayashi, Y., 1982: Space-time spectral analysis and its applications to atmospheric waves. *J. Meteor. Soc. Japan*, **60**, 156-171.

Holloway, J. L. Jr., 1958: Smoothing and filtering of time series and space fields. *Adv. Geophys.*, **4**, 351-389.

Jenkins, G. M., and D. G. Watts, 1968: *Spectral Analysis and Its Applications*. Holden-Day, 525 pp.

Kay, S. M., 1987: *Modern Spectral Estimation*. Prentice-Hall, Inc, Englewood Cliffs, New Jersey, Kay, S.M., 1988: Modern spectral estimation--Theory and application. Prentice-Hall, 543p.

Keppenne, C.L., and Ghil, M., 1992: Adaptive spectral analysis and prediction of the Southern Oscillation Index, J.Geophys. Res., 97, 20449-20554.

Keppenne, C.L., and Ghil, M., 1993: Adaptive filtering and prediction of noisy multivariate signals: an application to atmospheric angular momentum, Intl. J. Bifurcation and Chaos, in press.

Madden, R. A., and P. R. Julian, 1971: Detection of a 40-50 day oscillation in the zonal wind in the tropical Pacific. *J. Atmos. Sci*., **28**, 702-708.

Manly, B.F.J., 1986: Multivariate statistical methods. Chapman and Hall, 159 pp.

Marple, S. L. J., 1987: *Digital Spectral Analysis with Applications*. Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 492 pp.

Marple, S.L., 1987: Digital spectral analysis, with applications. Prentice-Hall, 492 pp.

Marsaglia, Zaman, and Tsang, 1990: Toward a universal random number generator, Stats & Prob Letters, 8, 35-39.

Naidu, P. S., 1995, Modern Spectrum Analysis of Time Series: Boca Raton, CRC Press, 399 p.

Nikias, C. L., and Petropulu, A. P., 1993, Higher-Order Spectra Analysis: A Nonlinear Signal Processing Framework: Prentice Hall Signal Processing Series: Englewood Cliffs, Prentice-Hall, 537 p.

North, G.R., Bell, T.L., Cahalan, R.F., and Moeng, F.J., 1982: Sampling errors in the estimation of empirical orthogonal functions, Mon. Wea. Rev., 110, 699-706.

Palus, M., and Dvorak, I., 1992: Singular-value decomposition in attractor reconstruction: Pitfalls and precautions, Physica D, 55, 221-234.

Penland, C., and Magorian, T., 1993: Prediction of Niño 3 sea surface temperatures using linear inverse modeling, J. Clim., 6, 1067-1076.

Penland, C., Ghil, M., and Weickmann, K., 1991: Adaptive filtering and maximum entropy spectra with application to changes in atmospheric angular momentum: J. Geophys. Res., 96, 22659-22671.

Percival, D.B., and Walden, A.T., 1993: Spectral analysis for physical applications--Multitaper and conventional univariate techniques. Cambridge University, 580 p.

Pratt, R.W., 1976: The interpretation of space-time spectral quantities. *J. Atmos. Sci*., **33**, 1060-1066.

Press, W.H., Flannery, B.P., Teukolsky, S.A., and Vetterling, W.T., 1989: Numerical recipes--The art of scientific computing (FORTRAN version). Cambridge University Press, 702 p.

Rasmusson, E.M., Wang, X., and Ropelewski, 1990: The biennial component of ENSO variability, J. Marine Sys., 1, 71-96.

Shäfer, J., 1979: A space-time analysis of tropospheric planetary waves in the Northern Hemisphere. *J. Atmos. Sci.*, **36**, 1117-1123.

Thomson, D.J., 1982: Spectrum estimation and harmonic analysis, Proc. IEEE, 70, 1055-1096.

Thomson, D.J., 1990a: Time series analysis of Holocene climate data, Phil. Trans. R. Soc. Lond A, 330, 601-616.

Thomson, D.J., 1990b: Quadratic-inverse spectrum estimates: applications to paleoclimatology, Phil. Trans. R. Soc. London A, 332, 593-597.

Thomson, D.J., and Chave, A.D., 1990: Jackknifed error estimates for spectra, coherences, and transfer functions, Chapter 2 in Haykin, S. (ed.), Advances in Spectrum Analysis. Prentice-Hall.

Tsonis, A.A., and Elsner, J.B., 1992: Oscillating global temperature, Nature, 356, 751.

Ulrych, T.J., and T.N. Bishop, 1975: Maximum entropy spectral analysis and autoregressive decomposition. *Rev. Geophys. and Space Physics*, **13**, 183-200.

Vautard, R., and Ghil, M., 1989: Singular spectrum analysis in nonlinear dynamics, with applications to paleoclimatic time series, Physica D, 35, 395-424.

Vautard, R., Yiou, P., and Ghil, M., 1992: Singular-spectrum analysis: A toolkit for short, noisy chaotic signals, Physica D, 58, 95-126.

Welch, P. D., 1967: USE OF FAST FOURIER TRANSFORM FOR ESTIMATION OF POWER SPECTRA - A METHOD BASED ON TIME AVERAGING OVER SHORT MODIFIED PERIODOGRAMS. *Ieee Transactions on Audio and Electroacoustics*, **AU15,** 70-73.

Yiou, P., Genthon, C., Ghil, M., Jouzel, J., LeTreut, H., Barnola, J.M., Lorius, C., and Korotkevitch, Y.N., 1991, High-frequency paleovariability in climate and CO2 levels from Vostok ice-core records: J. Geophys. Res., 96, 20365-20378.

Yiou, P., E. Baert and M. F. Loutre, 1996: Spectral Analysis of Climate Data. *Surveys in Geophys*, **17**, 619-663.

Source code for a Singular Spectrum Analysis Toolkit can be found at
"***http://www.atmos.ucla.edu/tcd/ssa/index.html***"