Hydrostatic Balance

Total state = basic state of the resting atmosphere + the perturbation of current atmosphere

\[ p(x, y, z, t) = p_0(z) + p'(x, y, z, t) \]

Vertical Momentum Equation (before scaling)

\[
\frac{Dw}{Dt} \left( \frac{u^2 + v^2}{a} \right) = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g - 2\Omega u \sin \phi + \nu \nabla^2 w \\
0 = \frac{\partial p}{\partial t} + \nabla (\rho \bar{u})
\]

The Terms

\[
\begin{align*}
\frac{Dw}{Dt} & \quad \text{vertical velocity} \\
\frac{u^2 + v^2}{a} & \quad \text{Curvature Terms} \\
-\frac{1}{\rho} \frac{\partial p}{\partial z} & \quad \text{pressure gradient} \\
g & \quad \text{apparent gravitational force} \\
-2\Omega u \sin \phi & \quad \text{coriolis force} \\
\nu \nabla^2 w & \quad \text{frictional force}
\end{align*}
\]

Decomposition

\[-\frac{1}{\rho} \frac{\partial p}{\partial z} = -\frac{1}{\rho_0 + \rho'} \frac{\partial (p_0 + p')}{\partial z} \approx -\frac{1}{\rho_0} \frac{\partial p_0}{\partial z} + \frac{1}{\rho_0} \frac{\partial p'}{\partial z} + \frac{p'}{p_0} g\]

After Decomposition

\[
\frac{Dw}{Dt} \left( \frac{u^2 + v^2}{a} \right) = -\frac{1}{\rho_0} \frac{\partial p_0}{\partial z} + \frac{1}{\rho_0} \frac{\partial p'}{\partial z} + \frac{p'}{p_0} g - g - 2\Omega u \cos \phi + \nu \nabla^2 w
\]

Scaling

\[
\frac{Dw}{Dt} \left( \frac{u^2 + v^2}{a} \right) = -\frac{1}{\rho_0} \frac{\partial p_0}{\partial z} + \frac{1}{\rho_0} \frac{\partial p'}{\partial z} + \frac{p'}{p_0} g - g - 2\Omega u \cos \phi + \nu \nabla^2 w \quad \left( \text{10}^{-7} \right)
\]

Scaling - Discarding Terms

\[
\frac{Dw}{Dt} \left( \frac{u^2 + v^2}{a} \right) = -\frac{1}{\rho_0} \frac{\partial p_0}{\partial z} + \frac{1}{\rho_0} \frac{\partial p'}{\partial z} + \frac{p'}{p_0} g - g - 2\Omega u \cos \phi + \nu \nabla^2 w \quad \left( \text{10}^{-7} \right)
\]

Vertical Equation of Motion

\[
g = -\frac{1}{\rho'} \frac{\partial p'}{\partial z}
\]

Continuity Equation

\[ \text{..image:: continuity_box.jpg} \]
Inflow of Mass from the Left

\[ -M_{\delta x} = \left( (\rho u)_0 - \frac{\partial (\rho u)}{\partial z} \frac{\delta x}{2} \right) \delta y \delta z \]

Inflow of Mass from the Right

\[ M_{\Lambda x} = \left( (\rho u)_0 - \frac{\partial (\rho u)}{\partial z} \frac{\delta x}{2} \right) \delta x \delta z \]

Net Mass Inflow

The net mass inflow per unit volume

\[ M_s = -\frac{\partial (\rho u)}{\partial x} \delta x \delta y \delta z = -\frac{\partial (\rho u)}{\partial x} \delta V \]

Local Density Change

\[ \frac{M_s + M_y + M_z}{\delta V} = \left( \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} \right) = \frac{\partial \rho}{\partial t} \]

\[ = -\nabla (\rho \bar{u}) \]

Mass Conservation

\[ \frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0 \]

\[ \implies \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \bar{u}) = 0 \]

Then, the total derivative is

\[ \frac{1}{\rho} \frac{D\rho}{Dt} + \nabla \bar{u} = 0 \]

Decomposition of Total Density

\[ \frac{1}{\rho} \frac{D\rho}{Dt} = \frac{1}{(\rho_0 + \rho')} \frac{D(\rho_0 + \rho')}{Dt} \approx \frac{1}{\rho_0} \left( 1 - \frac{\rho'}{\rho_0} \right) \frac{D(\rho_0 + \rho')}{Dt} \]

\[ \approx \frac{1}{\rho_0} \left( \frac{\partial \rho'}{\partial t} + \bar{u} \cdot \nabla \rho' \right) + \frac{\rho_0}{\rho_0} \frac{D\rho}{dz} \]

Scale Analysis

\[ \frac{1}{\rho_0} \left( \frac{\partial \rho'}{\partial t} + u \frac{\partial \rho'}{\partial x} + v \frac{\partial \rho'}{\partial y} + w \frac{\partial \rho'}{\partial z} \right) + \frac{\rho_0}{\rho_0} \frac{D\rho}{dz} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \]

Scaled Continuity Equation

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} + \frac{\rho_0}{\rho_0} \frac{D\rho}{dz} = 0 \]
Or, using the divergence operator

\[ \nabla \cdot (\rho \vec{u}) = 0 \]

**Thermodynamic Energy Equation**

**First Law of Thermodynamics**

Internal change in energy plus the work done by the air parcel is equal to the external energy input.

\[ c_v \frac{DT}{Dt} + p \frac{Da}{Dt} = J \]

\[ \alpha = \frac{1}{p} \]

- \( c_v \): specific heat at constant volume
- \( c_v T \): internal energy of the air parcel

**Ideal Gas Law**

\[ p = \rho RT \]

\[ pa = RT \]

\[ \frac{Dp}{Dt} (ap) = R \frac{DT}{Dt} = p \frac{Da}{Dt} + \alpha \frac{Dp}{Dt} = R \frac{DT}{Dt} \]

**Rewriting the First Law**

\[ c_p \frac{DT}{Dt} - \alpha \frac{Dp}{Dt} = J \]

dividing by \( T \)

\[ c_p \frac{D \ln T}{Dt} - R \frac{D \ln p}{Dt} = \frac{J}{T} \]