The motion of the atmosphere and ocean presents a formidable challenge to the climate modeler. First, there is no clear consensus about what level of detail is appropriate, and second, no matter what level of detail we choose, the equations are too difficult to solve algebraically, so we must resort to numerical solutions somewhat like our spreadsheet models. A typical model of the general circulation may have \(10^7\) variables and require \(10^7\) equations, which is too big to solve, and too big to interpret. What is needed is a smaller set of equations that retains the essential features of the full problem. In this model we will study a set of three coupled equations that will give us insight into the complexities of fluid flow.

The ideas we will study started with a brilliant piece of work by Barry Saltzman\(^1\). Saltzman, who worked at Yale, was studying the currents that arise in a beaker of water that is heated from below. To model this process, he produced a set of three variables and three equations for the way they change in time. When a beaker of water is heated from below, there is no motion for low heating rates, but as the heating rate is turned up, the water passes through a number of organized convection patterns and eventually the convection becomes vigorous and apparently random.

The Saltzman equations do not allow us to calculate earth’s climate. How could they? But the patterns that appear in the solutions to Saltzman’s equations should represent patterns of fluid motion in the full climate system.

Saltzman’s equations are:

\[
\begin{align*}
X(n+1) &= X(n) + dt(-\sigma X(n) + \sigma Y(n)) \\
Y(n+1) &= Y(n) + dt \left(-X(n)Z(n) + r X(n) - Y(n)\right) \tag{S-L equations} \\
Z(n+1) &= Z(n) + dt \left(X(n)Y(n) - bZ(n)\right).
\end{align*}
\]

Think of \(X, Y,\) and \(Z\) as being any three climate variables. We don’t care what they mean, just how they behave. We note several comforting features. Each equation has the form

\[
\text{rate of change of } X = \frac{X(n+1)-X(n)}{dt} = f(X(n), Y(n), Z(n)).
\]

The equations use values at the previous time step, \(n\), to calculate new values at the present time, \(n+1\), so it will be easy to calculate the time series using Excel. Each function on the right hand side contains a term like \(-X, -Y,\) or \(-Z\). Thus if \(X\) is large it will tend to decrease, and similarly for \(Y\) and \(Z\). These terms keep the system bounded.

The only new feature in the S-L equations is that two of the equations contain non-linear terms: \(XZ\) in the \(Y\)-equation and \(XY\) in the \(Z\)-equation. Here, as elsewhere, the presence of nonlinear terms completely changes the nature of the solution. In fact, the first thing you should ask about a system of equations is: are they linear? If they are, you can always solve them. If they are not, you are not smart enough to solve them, and probably no one you know is smart enough to solve them. So don’t waste time trying to solve. Go directly to Excel.
Edward Lorenz, at MIT, in another landmark paper, found properties of the Saltzman equations that were similar to properties of climate variables. Our investigation follows Lorenz’s ideas. First we notice parameters $\sigma$, $r$, and $b$. We anticipate that different patterns of motion will occur for different ranges of parameters, just as we found for the savings account model.

**Equilibrium Solutions**

If we express the S-L equations as a rate of change, as in the previous equation, and set the right hand sides to equal zero, we will find equilibrium states. For any choice of parameters, $(X, Y, Z) = (0, 0, 0)$ is an equilibrium solution. Further, if we set $X(n + 1) = X(n)$, as it must for equilibrium, we obtain

$$X = Y.$$ 

In a similar way for $Y$ and $Z$ we obtain

$$Z = r - 1;$$
$$X^2 = b(r - 1).$$

Therefore we have three equilibrium points:

<table>
<thead>
<tr>
<th></th>
<th>Equilibrium 1</th>
<th>Equilibrium 2</th>
<th>Equilibrium 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X$</td>
<td>0</td>
<td>$\sqrt{b(r - 1)}$</td>
<td>$-\sqrt{b(r - 1)}$</td>
</tr>
<tr>
<td>$Y$</td>
<td>0</td>
<td>$\sqrt{b(r - 1)}$</td>
<td>$-\sqrt{b(r - 1)}$</td>
</tr>
<tr>
<td>$Z$</td>
<td>0</td>
<td>$r - 1$</td>
<td>$r - 1$</td>
</tr>
</tbody>
</table>

Near Equilibrium 1, the values of $X$, $Y$, and $Z$ are all small so we can neglect any terms involving products of the variables, leaving a linearized version of the S-L equations:

- $dX/dt = -\sigma X + \sigma Y$
- $dY/dt = rX - Y$ (linearized S-L equations)
- $dZ/dt = -bZ$

The $Z$ equation does not involve $X$ or $Y$, so it can be solved separately: $Z(t) = Z(0)e^{-bt}$. If $b$ is positive, then $\lim_{t \to \infty} Z(t) = 0$. Eventually the trajectory will be restricted to the $XY$-plane.

The $X$ and $Y$ equations do not depend on $Z$, but they do depend on each other. The $X$ and $Y$ equations form a set of coupled linear differential equations and have solutions of the form $X(t) = c_1 \exp(\lambda_1 t) + c_2 \exp(\lambda_2 t)$, and similarly for $Y(t)$. Here the constants $c_1$ and $c_2$ come from the initial condition, and $\lambda_1$ and $\lambda_2$ are the roots of the characteristic equation

$$\begin{vmatrix} -\sigma - \lambda & \sigma \\ r & -1 - \lambda \end{vmatrix} = 0.$$ 

Solutions of this equation can be real or imaginary numbers depending on the values of the parameters $\sigma$ and $r$, so we expect classes of solutions with different behavior.
Additionally, we can write the linearized S-L equations in the form \( \frac{d\vec{X}}{dt} = M \vec{X} \) where

\[
\vec{X} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}, \quad M = \begin{bmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{bmatrix}.
\]

The trace of \( M \), or the sum of the diagonal elements, is equal to the divergence of the velocity vector:

\[
\nabla \cdot \frac{d\vec{X}}{dt} = -\sigma - 1 - b.
\]

If we insist that the parameters \( \sigma \) and \( b \) are positive, then the divergence of the velocity vector is always negative. Therefore, if we track an initial 3-dimensional volume \( V \), it will stretch in some direction and shrink in other directions, but the shrinking will dominate. As \( t \to \infty \), the volume \( V \) will map onto a space of dimension less than 3. In fact, trajectories follow flat surfaces consisting of many 2-dimensional leaves with gaps between them. Solutions are drawn to so-called “Lorenz attractors,” which are a feature of well-developed fluid flow.

What are Lorenz attractors? Let’s review a bit. The variables \( X \), \( Y \), and \( Z \) refer to three properties of the atmospheric and oceanic circulation. We can think of them as the intensity of the jet stream, the energy associated with tropospheric storms, and the energy carried by ocean currents. The parameters \( \sigma \), \( r \), and \( b \) represent properties of the atmosphere and ocean: such things as the solar heating, the albedos of ice, water, clouds, the rate of rotation of earth. There may be 50 parameters we could think of in a more detailed model of the climate. But for the simple model of Saltzman and Lorenz, the 50 dimensional variables combine to form three dimensionless parameters: \( \sigma \), \( r \), and \( b \). Saltzman and Lorenz were certainly thinking of Rayleigh’s 1916 treatment of thermal convection in a fluid heated from below, in which appear the dimensional quantities the acceleration of gravity, the coefficient of thermal expansion, the thermal conductivity, the viscosity of the fluid, and the dimensions of the container – all rolled up into a single dimensionless parameter.

No one thinks these equations make a climate model. But many believe that studying these equations can provide useful insights about the real climate. The solution to the S-L equations is a 1-dimensional path in a finite 3-dimensional box. Thinking geometrically, we say that for low values of the parameters \( \sigma \), \( r \), and \( b \) the solutions are fixed points. For somewhat larger values, periodic solutions appear. For even larger values, solutions will trace out a line \( (X(t), Y(t), Z(t)) \). The line starts at the initial point \( (X(0), Y(0), Z(0)) \), but, eventually, it will forget about where it started and from then on it will be constrained to some restricted region of the 3-dimensional box. We say the solution is drawn to the Lorenz attractor.

The Lorenz attractor has unusual geometric properties. It may have infinite length, but fit inside a finite box. It may have surfaces, stacked together line a deck of cards, or like the leaves in a head of cabbage. The panels in Figure 10.1 are an attempt to display the Lorenz attractor. They all come from the same calculation of a finite segment a typical trajectory. The purpose is to show what the attractor looks like, viewed from different directions.
Figure 10.1 A segment of a trajectory that traces out a type of Lorenz attractor set known as a *strange* attractor. The trajectory follows a three-dimensional path in a box, and these panels show what the path looks like as viewed from different directions. The parameter values were $\sigma = 10, b = 37, r = 4$.

From Figure 10.1 it is clear that attractor set occupies only a small fraction of $XYZ$-space. If we think of $XYZ$-space as the set of all conceivable climates, the attractor set consists of those climates that are consistent with the S-L equations. It could be that some points in $XYZ$-space correspond to clockwise and anticlockwise circulation around a low-pressure system. The attractor set includes the former but not the latter. There are certain types of motion that the atmosphere displays and other types it avoids.

The S-L equations describe mixing in a fluid. If you follow the paths of many nearby points defining a volume $V$, after a long time the points will be widely dispersed over some part of the attractor set. Figure 10.2 illustrates this idea with two trajectories. The figure shows two solutions that start close together. Their orbits track together for a while, and then rapidly diverge. Similarly, Figure 10.3 shows the trajectories of many points initially close together. The points occupy a region that is a square at first, and after 5,000 time steps has stretched out in one direction and squeezed together in another. But after 30,000 iterations, the initial points have spread throughout $XYZ$-space showing no sign that they were once close. This behavior is known as “sensitive dependence on initial conditions.”
**Figure 10.2** Two trajectories showing sensitive dependence on initial conditions. The two trajectories begin close together in the open circle. At the solid circle the trajectories suddenly begin to diverge and follow independent paths in the attractor set, shown by the solid arrows. The parameter values are $\sigma = 10, b = 4, r = 16$.

**Figure 10.3** Solutions of the Lorenz-Saltzman equations for about 100 points that start close together. The initial conditions are shown in black (zoomed in, upper left figure). After 5,000 steps, shown in green, the points are still close together but are squeezed into a line (zoomed in, lower right figure). After 30,000 steps, shown in blue, the points are scattered in $XY$ space. Predictability is lost at some time between 5,000 and 30,000 steps.
Consider the implications for predicting the state of the atmosphere or ocean based on its present values. Even if we had exact equations and could solve them without error, we cannot hope to avoid the consequences of small errors in the present state. The best we can do is to follow the trajectories of a small cloud of points around our estimate of the initial state. As long as the cloud remains small then the forecast model will have some skill. But no cloud of points remains a cloud for long. Once the cloud begins to break up, all forecasting skill is lost. The S-L equations set a limit to the predictability of sufficiently energetic fluid flow.

**Exercises**

10.1 Construct a model of the Saltzman-Lorenz equations.

10.2 By brute force, find conditions on the parameters \( \sigma, r \) and \( b \), for convergence to a fixed point.

10.3 Show the trajectories for cases with 0, 1, 2, and 3 positive eigenvalues of \( M \).

10.4 Construct an iterative solution. Plot the result as an animation of a moving point in the \( XY \)-plane.

10.5 For a single trajectory, save only values \( Z(n) \) that are local maxima, by which I mean \( Z(n) > Z(n - 1) \) and \( Z(n) > Z(n + 1) \). This will give you a sequence of \( Z \) values: \( \{Z_{\text{max}}(1), Z_{\text{max}}(2), \ldots\} \). Plot these points by considering the horizontal axis to be \( Z_{\text{max}}(n) \) and the vertical axis to be \( Z_{\text{max}}(n + 1) \). For example, if the \( Z \) values are

\[
Z = \{1, 3, 4, 6, 5, 3, 4, 5, 7, 9, 8, 9, 5, 1, 2, 3, 4, 7, 5, 3, 2, \ldots\}
\]

then the \( Z_{\text{max}} \) sequence would contain the underlined values and you would plot the pairs (6, 9), (9, 9), (9,7), and so on.

**References**


Hints

10.5 Here is a Visual Basic routine for extracting the local maxima from $Z(n)$:

Sub Lorenz20()
Dim sh As Excel.Worksheet
Set sh = Application.ActiveSheet
nt = sh.Cells(8, 1)
dt = 0.002
n = 11
dx = 0.01
jrow = 0
For jj = 1 To n
    For j = 1 To n
        jrow = jrow + 1
        xc = 0.3
        yc = -0.2
        x0 = xc + jj * dx
        y0 = yc + j * dx
        z0 = 2 * dx
        s = 20
        r = 36
        b = 4
        For k = 1 To nt
            x1 = x0 + dt * (-s * x0 + s * y0)
            y1 = y0 + dt * (-x0 * z0 + r * x0 - y0)
            z1 = z0 + dt * (x0 * y0 - b * z0)
            x0 = x1
            y0 = y1
            z0 = z1
        Next k
Next jj
Next n
Next jj
Next k
sh.Cells(10 + jrow, 1) = x0
sh.Cells(10 + jrow, 2) = y0
sh.Cells(10 + jrow, 3) = z0
Next j
Next jj
End Sub