8.1 Aliasing

Because the DFT is based on sampling a continuous function at a finite set of equally-spaced points \( j \Delta t \), many different \( L \)-periodic functions can have the same DFT. In fact, different sinusoids can have the same DFT, an ambiguity called aliasing. In general, consider harmonic \( M \) sampled on the grid \( t_j = (j - 1) \Delta t \) where \( j = 1, ..., N \) and \( \Delta t = L/N \):

\[
\exp(i \omega_M t_j) = \exp \left[ \frac{2 \pi M}{L} (j - 1) \frac{L}{N} \right] = \exp \left[ 2 \pi i \frac{M(j - 1)}{N} \right]
\]

Harmonics \( M + qN, \quad q = \pm 1, \pm 2, ... \) also have exactly the same values at the grid points, since

\[
\exp(i \omega_{M+qN} t_j) = \exp \left[ 2 \pi i \frac{(M + qN)(j - 1)}{N} \right]
= \exp \left[ 2 \pi i \frac{M(j - 1)}{N} \right] \cdot \exp \left[ 2 \pi i q(j - 1) \right]
= \exp(i \omega_M t_j)
\]

Thus there is always ambiguity in whether one is looking at a smooth, adequately sampled signal, or a highly oscillatory, poorly sampled signal. Our choice of how to assign harmonics \( M \) to the different components \( m \) of the DFT was based on assuming the signal is smooth. A signal composed of highly oscillatory harmonics that alias to a low harmonic \( M \) on the given grid will just add to the true signal from that harmonic. For instance DFT element \( m = 1 \) includes the signal not only from harmonic \( M = 0 \) but also from \( M = \pm N, \pm 2N, ... \), as shown in the left panel of Fig. 1.

The Nyquist frequency \( \omega_{N/2} = \pi/\Delta t \) is the maximum frequency that is unambiguously detectable on the grid. It corresponds to a \( (1, -1, 1, -1, ...) \) oscillation on the grid of period \( 2 \Delta t \) (right panel of Fig. 1).
8.2 Matrix form of DFT/IDFT; Parseval’s Thm

The DFT and IDFT can be expressed in matrix form. If \( u \) is the vectors of gridpoint values \( u_j \), then:

\[
\hat{u} = \text{DFT}(u) = N^{1/2} Fu,
\]

\[
u = \text{IDFT}(\hat{u}) = N^{-1/2} F^\dagger \hat{u},
\]

where \( \hat{u} \) is the DFT of \( u \), the elements of the DFT matrix \( F \) are

\[
F_{mj} = N^{-1/2} \exp(-2\pi i (m-1)(j-1)/N),
\]

and \( F^\dagger \) is the conjugate transpose of \( F \).

We showed above that the IDFT is the inverse of the DFT, so

\[
u = N^{-1/2} F^{-1} \hat{u} \Rightarrow F^{-1} = F^\dagger.
\]

That is, \( F \) is a unitary matrix. This gives an easy derivation of Parseval’s theorem

\[
\sum_{m=1}^{N} |(\hat{u}_m/N)^2| = \hat{u}^\dagger \hat{u}/N^2
\]

\[
= u^\dagger F^\dagger Fu/N
\]

\[
= u^\dagger u/N
\]

\[
= N^{-1} \sum_{j=1}^{N} |u_j^2|.
\]

That is, the sum of the squares of the approximate Fourier coefficients \( \hat{u}_m/N \) is equal to the average power or squared amplitude of the time series \( u_j \). We
interpret Parseval’s theorem as a partitioning of the power into contributions from each harmonic or wavenumber; this is very useful for interpretation of data.

8.3 Key things to remember about the DFT

Matlab DFT: \( \texttt{uhat} = \text{fft}(\texttt{u}) \); inverse DFT: \( \texttt{u} = \text{ifft}(\texttt{uhat}) \).

Will calculate the DFT or inverse DFT using the ‘fast’ algorithm if the data length is \( N = 2^p3^q5^r \). For other \( N \), it will take \( O(N^2) \) flops and go much slower if \( N \) is large.

**Assumes periodic input** : \( u_{N+1} = u_1 \) (discontinuities between the endpoints can create unintended artifacts)

**Relation to Fourier series** If \( u \) is sampled from a continuous periodic function \( u(t) \), \( \texttt{uhat}/N \) gives an estimate of its complex Fourier series coefficients \( c_M \):

\[
\hat{u}_m/N \approx c_M, M = m-1 \ (1 \leq m \leq N/2) \text{ or } m-1-N \ (N/2+1 \leq m \leq N).
\]

For smooth functions \( u(t) \) and low-order harmonics, this approximation is extremely accurate. Parseval’s theorem partitions the power in \( u \) into the Fourier modes or harmonics in its DFT.

**Account for the shift** between the indices \( m \) and the corresponding Fourier harmonics \( M_m \). In Matlab, define the index vector of harmonics \( M = [0:(N/2-1) -N/2:-1] \) and the frequencies \( \omega_M = 2\pi M/L \) (or wave numbers \( k = 2\pi M/L \) in a problem in which position \( x \) is the independent variable).

\( m=1 \) coefficient of \( \texttt{uhat} \) is \( N \) times the mean of \( u \) (easily proved from DFT definition).

**DFT is complex-valued** If \( u \) is real, the DFT coefficients for Fourier modes \( M \) and \( -M \) are complex conjugates (easily proved from DFT definition).

\( x \) derivative of spatially periodic function Matlab: \( \texttt{dudx} = \text{real}(\text{ifft}(1i*k*\texttt{fft}(u))) \);