Singularly Perturbed Boundary-Value Problems and Boundary Layers

Example S2: Find an approximate solution to
\[
\varepsilon y'' + \frac{\partial y}{\partial \varepsilon} + y' = 1, \quad y(0) = 1, y(1) = 0, \quad 0 < \varepsilon \ll 1. \tag{S2}
\]

This is a singular perturbation problem since the unperturbed problem is a 1st-order ODE and the perturbed problem in a 2nd-order ODE with an extra linearly independent solution.

As always, we start by considering the unperturbed ODE
\[
y_0' = 1 \quad \Rightarrow \quad y_0(x) = x + c_0
\]
Regardless of \(c_0\), this solution cannot satisfy both BCs. However, it will still be very useful!

This BVP is a nice learning case because it has a simple exact solution of the form
\[
y_{ex}(x; \varepsilon) = x + c_0 + c_1 e^{-x/\varepsilon}
\]
Define \(\delta = e^{x/\varepsilon}\). To satisfy the BCs,

\begin{align*}
(BC1) & \quad 1 = y(0) = c_0 + c_1 \\
(BC2) & \quad 0 = y(1) = 1 + c_0 + c_1 \delta \quad \Rightarrow \quad c_1 = \frac{2}{1-\delta}, \quad c_0 = -\frac{1+\delta}{1-\delta}
\end{align*}

For \(0 < \varepsilon \ll 1\), \(\delta\) is extremely small, so \(c_1 \approx 2, c_0 \approx -1\), and
\[
y(x; \varepsilon) = x - 1 + 2 e^{-x/\varepsilon} \approx \begin{cases} -1 + 2 e^{-x/\varepsilon}, & 0 \leq x \ll 1 \quad \text{‘inner’ solution} \\
x - 1, & x \gg \varepsilon \quad \text{‘outer’ solution} \end{cases}
\]

Adjacent to the boundary \(x = 0\), \(y(x)\) changes rapidly across a boundary layer of thickness \(O(\varepsilon)\).

As an interesting aside, were \(\varepsilon < 0\), but still small, then \(\delta >> 1\), \(c_1 \approx -2 e^{-1/\varepsilon}, c_0 \approx 1\), and
\[
y(x; \varepsilon) = x + 1 - 2 e^{-(x-1)/\varepsilon}; \text{ in this case the boundary layer would be next to } x = 1 \text{ instead!}
\]

Here’s how we can think about this problem using the method of dominant balance. In the boundary layer, the dominant balance in the ODE is \(A\) (the highest derivative, proportional to \(\varepsilon\)) vs. \(B\) (the highest derivative in the unperturbed ODE). This yields an approximate ‘inner’ solution on which we can apply the \(x = 0\) BC:

**Inner problem \((x = O(\varepsilon))\):** \(\varepsilon y'' + y' = 0, \ y(0) = 1 \quad \Rightarrow \quad y_{inner}(x) = 1 - c_1 + c_1 e^{-x/\varepsilon} \tag{9.2}\)

Outside the boundary layer, the dominant balance is \(B\) vs. \(C\). This gives the unperturbed ODE, and the \(x = 1\) BC is in this region:

**Outer problem \((x >> O(\varepsilon))\):** \(y' = 1, \ y(1) = 0 \quad \Rightarrow \quad y_{outer}(x) = x - 1 \tag{9.3}\)
For the inner solution (9.2) to match the outer solution (9.3) across the edge of the boundary layer,
\[
\lim_{x/\epsilon \to \infty} y_{\text{inner}}(x) = \lim_{x \to 0} y_{\text{outer}}(x)
\]
\[1 - c_1 = 0 - 1 \implies c_1 = 2\]

In this way, we have recovered the approximate solution (9.1) to the exact problem. This solution is asymptotically accurate as \(\epsilon \to 0\) in both the ‘outer’ sense (with \(x\) fixed, in which case the solution asymptotes to the outer solution), and the ‘inner’ sense (with \(x/\epsilon\) fixed, in which case the solution asymptotes to the inner solution). The \textit{uniformly valid} approximation \(y_{\text{uni}}(x; \epsilon) = x - 1 + 2e^{-x/\epsilon}\) of (9.1) encompasses both the inner and outer solutions in their respective regimes of validity. The inner, outer and exact solutions are plotted in Fig. 9.1 for \(\epsilon = 0.1\).

**Fig 9.1.** Asymptotic solution of (S2) showing boundary layer next to \(x = 0\).

**Example S3:** Approximately solve the singularly perturbed BVP
\[
-\epsilon y'' + y' + y^2 = 0, \quad y(0) = 1, \quad y(1) = 0, \quad 0 < \epsilon \ll 1 \tag{S3}
\]

This BVP can be derived as the steady state concentration of a chemical with concentration \(y(x)\) with nondimensional horizontal diffusivity \(\epsilon\) flowing in a tube from \(x = 0\), where its concentration is 1, to \(x = 1\) with speed 2, while reacting with itself at a rate \(y^2\). At \(x = 1\), the chemical is totally removed from the tube so \(y(1) = 0\).

One obvious new feature is the nonlinear term, which prevents the BVP from being solved exactly. A more subtle change from (S2) is the minus sign on the \(\epsilon\). As with (S2) we follow the

\textit{Method for singularly perturbed ODEs with boundary layers:}
(1) Look for an outer solution with dominant balance $B$ vs. $C$
(2) Look for an inner solution with dominant balance $A$ vs. $B$, applying relevant BCs.
(3) Asymptotically match the solutions and apply remaining BCs.

Outer (unperturbed) problem:
$$2y' + y^2 = 0 \Rightarrow -\frac{2 \, dy}{y^2} = dx \Rightarrow \frac{2}{y} = x - x_0 \Rightarrow y_{outer}(x) = \frac{2}{x - x_0}$$

By appropriate choice of the constant of integration $x_0$, this solution can be made to satisfy one BC but not both, and it is not yet obvious which BC to impose. This dominant balance of $B$ vs. $C$ is consistent, since $A = -\varepsilon y''_{outer} = O(\varepsilon) << B, C = O(1)$.

Inner problem:

For $A$ to be as large as either of $B$ or $C$, $y$ must vary by a large fraction of itself over a distance $L \ll 1$ (which we must determine). The outer solution and its mismatch with the BCs are both $O(1)$, so we assume that $y$ is $O(1)$ and scale all three terms. In this case,
$$A = \varepsilon y'' = O(\varepsilon/L^2), \quad B = 2y' = O(1/L), \quad C = y^2 = O(1)$$
Thus $B >> C$, so the dominant balance must be $A$ vs. $B$, which implies the scaling $L = \varepsilon$ and the inner problem
$$-\varepsilon y'' + 2y' = 0 \Rightarrow y_{inner}(x) = c_0 + c_1 e^{2x/\varepsilon}$$

The inner solution increases exponentially with lengthscale $L = \varepsilon$ to the right, so there is no way to use it to match the left BC0 at $x = 0$ to the more slowly varying outer solution. However, it can be used to match the right BC1 at $x = 1$ by setting
$$0 = y_{inner}(1) = c_0 + c_1 e^{2/\varepsilon} \Rightarrow c_1 e^{2/\varepsilon} = -c_0 \Rightarrow y_{inner}(x) = c_0 \left\{1 - e^{2(1-x)/\varepsilon}\right\}, \quad 1 - x = O(\varepsilon)$$
This represents a boundary layer adjacent to $x = 1$, like the $\varepsilon < 0$ case of (S2).

Matching outer vs. inner and applying BC0:

The outer solution must be valid for $0 \leq x < 1 - O(\varepsilon)$, in particular at $x = 0$. Hence we apply BC0:
$$1 = y_{outer}(0) = \frac{2}{0 - x_0} \Rightarrow x_0 = -2 \Rightarrow y_{outer}(x) = \frac{2}{x + 2}$$

Lastly, we must match with the inner solution:
$$\lim_{(1-x)/\varepsilon \to \infty} y_{inner}(x) = \lim_{x \to 1} y_{outer}(x) \Rightarrow c_0 = 2/3$$

We conclude that
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\[ y(x; \varepsilon) \sim \begin{cases} 
\frac{2}{3} \{1 - e^{2(x-1)/\varepsilon}\}, & 0 \leq 1 - x \ll 1 \quad \text{‘inner’ solution} \\
\frac{2}{x+2} & 1 - x \gg \varepsilon \quad \text{‘outer’ solution}
\end{cases} \]

for \( \varepsilon \ll 1 \).

Lastly, by regarding the inner solution as a correction term to the outer solution within the boundary layer \( 1 - x = O(\varepsilon) \), we can combine these two solution regimes into a uniformly valid solution across all \( x \), both inside and outside the boundary layer:

\[ y(x) \sim y_{\text{unif}}(x) = y_{\text{outer}}(x) + y_{\text{inner}}(x) - y_{\text{outer}}(1) = \frac{2}{x+2} - \frac{2}{3} e^{2(x-1)/\varepsilon}, \quad \varepsilon \ll 1 \]

Fig. 9.2 shows the inner, outer and uniformly valid asymptotic approximations to the solution for \( \varepsilon = 0.1 \), clearly showing the boundary layer structure and how the uniformly valid solution captures both solution regimes.

![Fig. 9.2: Asymptotic solution of (S3) showing boundary layer next to \( x = 1 \).](image)