As a next example, let’s use regular perturbation theory to solve a problem introduced in Lecture 1:

Find the nonlinear correction to the period $T(\epsilon)$ of a pendulum governed by

$$\frac{d^2\theta}{dt^2} + \sin\theta = 0, \quad \theta(0) = \epsilon \ll 1, \frac{d\theta}{dt}(0) = 0.$$ (P)

We’ve nondimensionalized time by multiplying it by the linear pendulum frequency $(g/L)^{1/2}$; this changes the coefficient of $\sin \theta$ from $g/L$ to 1.

It is always good to anticipate the expected behavior of the solution. The linear ($\epsilon = 0$) period is $T_0 = 2\pi$. The pendulum will have the same period if started from $\theta(0) = -\epsilon$ as from $\theta(0) = \epsilon$; this suggests the period is an even function of $\epsilon$. The simplest functional form consistent with these constraints is $T(\epsilon) = 2\pi + T_2 \epsilon^2 + O(\epsilon^4)$ for some unknown $T_2$.

We’ll use two approaches. For both, we note that the pendulum period is composed of four equal parts, starting with the clockwise downswing of $\theta$ from $\epsilon$ to 0, followed by a clockwise upswing, a counterclockwise downswing and a counterclockwise upswing. Thus it suffices to calculate the time $t = T/4$ when $\theta(t)$ first crosses zero. The first is a straightforward ‘brute force’ in which we use a perturbation series to solve the initial value problem (P), and the second will be to use potential theory from physics to find an integral expression for $T/4$, then use a perturbation approach to calculate this integral.

**Approach 1:** Perturbation series solution to IVP

Since we expect that $\theta(t)$ will oscillate between $\pm \epsilon$, we substitute into (P) a series of the form $\theta(t) = \epsilon \theta_1(t) + \epsilon^2 \theta_2(t) + \epsilon^3 \theta_3(t) \ldots$ With 20/20 hindsight, we keep terms through $O(\epsilon^3)$:

$$0 = \theta''(t) + \sin\theta = \epsilon \theta_1''(t) + \epsilon^2 \theta_2''(t) + \epsilon^3 \theta_3''(t) \ldots + \sin(\epsilon \theta_1(t) + \epsilon^2 \theta_2(t) + \epsilon^3 \theta_3(t) \ldots)$$ (5.1)

$$\theta(0) = \epsilon = 1 \epsilon + 0 \epsilon^2 + 0 \epsilon^3 \ldots, \quad \theta'(0) = 0 = 0 \epsilon + 0 \epsilon^2 + 0 \epsilon^3 \ldots$$ (5.2)

To separate the $\sin \theta$ term into powers of $\epsilon$, we use a Taylor series expansion:

$$\sin \theta = \theta - \theta^3/6 \ldots = \epsilon \theta_1(t) + \epsilon^2 \theta_2(t) + \epsilon^3 \theta_3(t) \ldots - (\epsilon \theta_1(t) \ldots)^3 / 6 \ldots$$

Substituting into (5.1) and (5.2) and separating powers of $\epsilon$:

$\epsilon^1$: $0 = \theta_1''(t) + \theta_1, \quad \theta_1(0) = 1, \theta_1'(0) = 0 \Rightarrow \theta_1(t) = \cos(t).$ (linear pendulum motion)

$\epsilon^2$: $0 = \theta_2''(t) + \theta_2, \quad \theta_2(0) = \theta_2'(0) = 0 \Rightarrow \theta_2(t) = 0.$
\[ \varepsilon^3: \quad \theta''_3(t) + \theta_3 = \frac{\theta_3^3}{6} = \frac{1}{6} \cos^3 t, \quad \theta_3(0) = \theta'_3(0) = 0 \]  

(5.3)

This ODE can be solved using a Laplace transform, using contour integration to invert the transform. This is an interesting problem left as a homework exercise; its solution is

\[ \theta_3(t) = \frac{1}{16} t \sin t + \frac{1}{192} (\cos t - \cos 3t) \]

Hence

\[ \theta(t) = \varepsilon \theta_1(t) + \varepsilon^3 \theta_3(t) \ldots = \varepsilon \cos t + \varepsilon^3 \left\{ \frac{1}{16} t \sin t + \frac{1}{192} (\cos t - \cos 3t) \right\} \ldots \]  

(5.4)

The script `pendulum.m` on the class web page compares this perturbation series to an ‘exact’ numerical solution of the full nonlinear pendulum equation using Matlab’s ode45 solver. Fig. 5.1 plots the solutions for two linear periods \(0 < t/2\pi < 2\) with \(\varepsilon = 0.5\) (top panel) and \(\varepsilon = 1\) (bottom panel). Even at these large values of \(\varepsilon\), the \(O(\varepsilon^3)\) correction produces quite an accurate approximation to \(\theta(t)\) by capturing the slow drift of the nonlinear solution from the linear solution.

![Graph comparing exact and perturbation series solutions](image.png)

Fig. 5.1: Comparison of ‘exact’ and perturbation series solutions to the nonlinear pendulum problem

To find the time \(T/4 = \pi/2 + T_2 \varepsilon^2/4 + O(\varepsilon^4)\) at which \(\theta(t)\) first crosses zero, we substitute this perturbation series into (5.4):
Thus (5.4) can be rewritten without the secular term:

\[
0 = \theta \left( \frac{T}{4} \right) = \varepsilon \cos \left( \frac{\pi}{2} + \varepsilon^2 \frac{T_2}{4} \ldots \right) + \varepsilon^3 \left\{ \frac{1}{16} \left( \frac{\pi}{2} + \varepsilon^2 \frac{T_2}{4} \ldots \right) \sin \left( \frac{\pi}{2} + \varepsilon^2 \frac{T_2}{4} \ldots \right) + \frac{1}{192} \left[ \cos \left( \frac{\pi}{2} + \varepsilon^2 \frac{T_2}{4} \ldots \right) - \cos \left( \frac{\pi}{2} + \varepsilon^2 \frac{T_2}{4} \ldots \right) \right] \right\} \ldots
\]

\[
= -\varepsilon \sin \left( \frac{\varepsilon^2 T_2}{4} \ldots \right) + \varepsilon^3 \left\{ \frac{1}{16} \left( \frac{\pi}{2} + \varepsilon^2 \frac{T_2}{4} \ldots \right) \cos \left( \varepsilon^2 \frac{T_2}{4} \ldots \right) + \frac{1}{192} \left[ -\sin \left( \varepsilon^2 \frac{T_2}{4} \ldots \right) - \sin \left( 3\varepsilon^2 \frac{T_2}{4} \ldots \right) \right] \right\} \ldots
\]

\[
= -\varepsilon^3 \frac{T_2}{4} \ldots + \varepsilon^3 \left\{ \frac{1}{16} \left( \frac{\pi}{2} \ldots \right) \right\} \ldots \Rightarrow T_2 = \frac{\pi}{8}
\]

Thus

\[
T(\varepsilon) = 2\pi + T_2 \varepsilon^2 \ldots = 2\pi \left( 1 + \frac{\varepsilon^2}{16} \ldots \right)
\]

The period lengthens slightly as the amplitude gets larger. Even for a swing amplitude of 10 degrees (\(\varepsilon = \pi/18\) radians), the predicted period is only 0.1% longer than the linear limit, which is why pendulums are useful timekeeping devices.

The O(\(\varepsilon^3\)) term in (5.4) has a first part which grows linearly with \(t\) and is therefore called a secular term. At very large times \(t = O(\varepsilon^{-2})\), this term becomes as large as the O(\(\varepsilon\)) term and the perturbation series loses validity. However, we can recognize this term as the phase drift due to the nonlinear pendulum oscillating slightly slower than the linear pendulum:

\[
\cos \left( \frac{2\pi t}{T} \right) = \cos \left( \frac{t}{1 + \varepsilon^2/16 \ldots} \right) = \cos \left( t \left[ 1 - \varepsilon^2/16 \ldots \right] \right) = \cos \left( t - \varepsilon^2 t/16 \ldots + \delta t \right)
\]

\[
= \cos t + \delta t \frac{d}{dt} (\cos t) \ldots = \cos t + \frac{\varepsilon^3}{16} t \sin t \ldots
\]

Thus (5.4) can be rewritten without the secular term:

\[
\theta(t) = \varepsilon \cos(2\pi t/T) + \frac{\varepsilon^3}{192} \left( \cos(2\pi t/T) - \cos(6\pi t/T) \right) \ldots (5.5)
\]

The method of multiple scales is a more advanced perturbation method to be discussed later which systematically introduces slow time or space scales to remove secular terms in a perturbation series.