We first note that this is a regular perturbation problem (why?) and naively try substituting a perturbation series of the form \( r(\varepsilon) = r_0 + \varepsilon r_1 + \varepsilon^2 r_2 \ldots \) into (2.3). Note that
\[
\begin{align*}
\varepsilon^3 r^3 &= \left( r_0 + \varepsilon r_1 + \varepsilon^2 r_2 \ldots \right) \left( r_0 + \varepsilon r_1 + \varepsilon^2 r_2 \ldots \right) \left( r_0 + \varepsilon r_1 + \varepsilon^2 r_2 \ldots \right) \\
&= r_0^3 + \varepsilon \left( 3r_0^2 r_1 + \varepsilon r_1 \right) + \varepsilon^2 \left( 3r_0 r_1^2 + 3r_0^2 r_2 \right) + \ldots \\
&= r_0^3 + \varepsilon \left( 3r_0^2 r_1 + \varepsilon r_1 \right) + \varepsilon^2 \left( 3r_0 r_1^2 + 3r_0^2 r_2 \right) + \ldots
\end{align*}
\]
so
\[
0 = r^3 + r^2 - \varepsilon
\]
\[
= r_0^3 + \varepsilon \left( 3r_0^2 r_1 + \varepsilon r_1 \right) + \varepsilon^2 \left( 3r_0 r_1^2 + 3r_0^2 r_2 \right) + \ldots + r_0^2 + \varepsilon \left( 2r_0 r_1 \right) + \varepsilon^2 \left( r_1^2 + 2r_0 r_2 \right) + \ldots - \varepsilon
\]
Collecting powers of \( \varepsilon \),
\[
\begin{align*}
\varepsilon^0: \quad 0 &= r_0^3 + r_0^2 \quad \Rightarrow \quad r_0' = -1, \quad r_0'' = 0. \quad \text{(unperturbed problem)} \\
\varepsilon^1: \quad 0 &= 3r_0^2 r_1 + 2r_0 r_1 - 1 \quad \Rightarrow \quad r_1 \left( 3r_0^2 + 2r_0 \right) = 1.
\end{align*}
\]
For \( r_0' = -1 \), this implies \( r_1' = \left( 3r_0^2 + 2r_0 \right)^{-1} = 1 \). We can continue the perturbation series for this case.
For \( r_0'' = 0 \), \( 3r_0^2 + 2r_0 = 0 \), so the equation for \( r_1 \) is inconsistent!
This implies that we need to use a different power \( \varepsilon^q \), as we’d anticipated.
\[
\begin{align*}
\varepsilon^2: \quad 0 &= 3r_0^2 r_1^2 + 3r_0^2 r_2 + r_1^2 + 2r_0 r_2 \\
&\text{Substituting } r_0 = -1 \text{ and } r_1 = 1 \text{ for root I:} \\
0 &= 3(-1)^2 + 3(-1)^2 r_2 + (-1)^2 + 2(-1)r_2 \quad \Rightarrow \quad r_2' = 2 \\
&\quad \Rightarrow \quad r'(\varepsilon) = -1 + \varepsilon + 2\varepsilon^2 \ldots \quad (3.1)
\end{align*}
\]
For the other two roots, since \( r_0=0 \) we try a series of the form \( r(\varepsilon) = \varepsilon^q r_1 + \varepsilon^{2q} r_2 \ldots \) Thus \( r^3 = O(\varepsilon^{3q}) \) and \( r^2 = O(\varepsilon^{2q}) \), so
\[
0 = r^3 + r^2 - \varepsilon = O(\varepsilon^{3q}) + O(\varepsilon^{2q}) - \varepsilon
\]
The only way to balance the \( \varepsilon \) is to make \( 2q = 1 \) (taking \( 3q = 1 \) would lead to an unbalanced \( O(\varepsilon^{3q}) \) term. Thus, as we suspected from the graph, we must choose \( q = 1/2 \). Thus
\[
\begin{align*}
0 &= r^3 + r^2 - \varepsilon \\
&= \varepsilon^{3/2} r_1^3 + \ldots + \varepsilon^{1/2} r_1^2 + \varepsilon^{3/2} (2r_0 r_2) + \ldots - \varepsilon
\end{align*}
\]
Collecting powers of \( \varepsilon \),
\[
\begin{align*}
\varepsilon^1: \quad 0 &= r_1^2 - 1 \quad \Rightarrow \quad r_1'' = \pm 1
\end{align*}
\]
\[ \epsilon^{3/2}: \quad 0 = r_1^3 + 2r_1r_2 \quad \Rightarrow \quad r_2^{\text{II,III}} = -\frac{r_1^2}{2} = -\frac{1}{2} \]
\[ \Rightarrow \quad r_2^{\text{II,III}}(\epsilon) = \pm \epsilon^{1/2} - \epsilon / 2 \cdots \] (3.2)

Fig. 3.1 plots the first two partial sums of the perturbation series for the three roots vs. \( \epsilon \) for the range \( 0 < \epsilon < 0.15 \), in which they are all real. They are compared with exact roots computed using the Matlab \texttt{roots} function. In this case, the perturbation series start losing accuracy for \( \epsilon \approx 0.1 \).

![Pert series for roots of \( r^3 + r^2 \epsilon \)](image)

Fig. 3.1: Convergence of perturbation series (3.1) and (3.2) for the three roots of the cubic.