Regular Perturbation Series

The solution of a regular perturbation problem is usually a smooth function of some power \( q \) of \( \varepsilon \) (e. g. \( \varepsilon, \varepsilon^2 \) or \( \varepsilon^{1/2} \)) near \( \varepsilon = 0 \). In this case, we can expand the solution in a Taylor series in \( \varepsilon^q \), which we call a perturbation series:

\[
y(x, \varepsilon) = y_0(x) + \varepsilon^q y_1(x) + \varepsilon^{2q} y_2(x) \ldots
\]  

(2.1)

The function \( y_0(x) \) (the solution of the \( \varepsilon = 0 \) problem) and the unknown functions \( y_1(x), y_2(x), \ldots \) are found by substituting (2.1) into the original problem (including any boundary conditions), separating the terms according to powers of \( \varepsilon^q \), and satisfying the successive problems at each order in \( \varepsilon \). The exponent \( q \) is usually 1, but when the choice of \( q \) is unclear, it can be made by substituting the proposed solution into the original equation and seeing what \( q \) is required to balance the perturbation terms (the higher powers of \( \varepsilon \)) in the equation. Like a normal Taylor series, the perturbation series is unique; if a series in powers of \( \varepsilon \) is assumed for \( y(x, \varepsilon) \), but \( y(x, \varepsilon) \) could actually be written as a series in powers of \( \varepsilon^2 \), then the coefficients of the odd powers of \( \varepsilon \) will all turn out to be identically zero. However, if \( y(x, \varepsilon) \) really were a series in powers of \( \varepsilon^{1/2} \), then our assumed series in powers of \( \varepsilon \) would be inconsistent. That is, we would find some orders of \( \varepsilon \) gave unbalanced equations where a nonzero function must equal zero. This would indicate that \( q \) should be changed to remove these imbalances.

Example (R1)

Find perturbation series for the roots of

\[
p(r; \varepsilon) = r^2 + \varepsilon r - 1 = 0, \quad \varepsilon \ll 1
\]

Note that for this problem, the two roots can be found exactly for any \( \varepsilon \) using the quadratic formula:

\[
r(\varepsilon) = \frac{-\varepsilon \pm (4 + \varepsilon^2)^{1/2}}{2}
\]  

(2.2)

so this is just a pedagogical exercise to get us started. We can see that the two roots are smooth functions of \( \varepsilon \) for small \( \varepsilon \) and we can anticipate from (2.2) that the perturbation series will be the Taylor series in \( \varepsilon \) that we could derive from (2.2), and that because \( r(\varepsilon) \) are analytic functions of the complex variable \( \varepsilon \) out to the nearest singularities (branch points at \( \varepsilon = \pm 2i \)), these series will have a radius of convergence of 2, i. e. they will converge for \( |\varepsilon| < 2 \). Let’s see if this really works!

Let us try a perturbation series of the form
\[ r(\varepsilon) = r_0 + \varepsilon r_1 + \varepsilon^2 r_2 \ldots \]

and substitute into the polynomial:

\[
0 = r^2 + \varepsilon r - 1 = (r_0 + \varepsilon r_1 + \varepsilon^2 r_2 \ldots)^2 + \varepsilon(r_0 + \varepsilon r_1 + \varepsilon^2 r_2 \ldots) - 1 \\
= (r_0^2 - 1) + \varepsilon(2r_0 r_1 + r_0) + \varepsilon^2(2r_0 r_2 + r_1^2 + r_1) \ldots
\]

For this to hold for arbitrary small values of \( \varepsilon \), coefficients of all powers of \( \varepsilon \) must vanish:

\[ \varepsilon^0: \quad 0 = r_0^2 - 1 \quad \Rightarrow \quad r_0^\pm = \pm 1 \]

\[ \varepsilon^1: \quad 0 = 2r_0 r_1 + r_0 \quad \Rightarrow \quad r_1^\pm = -\frac{1}{2} \]

\[ \varepsilon^2: \quad 0 = 2r_0 r_2 + r_1^2 + r_1 \quad \Rightarrow \quad r_2^\pm = -\frac{r_1^2 + r_1}{2r_0} = -\frac{-1/4}{\pm 2} = \pm \frac{1}{8} \]

so we obtain the perturbation series for the two roots:

\[ r^+(\varepsilon) = 1 - \frac{1}{2} \varepsilon + \frac{1}{8} \varepsilon^2 \ldots \]

\[ r^-(\varepsilon) = -1 - \frac{1}{2} \varepsilon - \frac{1}{8} \varepsilon^2 \ldots \]

The series for \( r^-(\varepsilon) \) is indeed the same as the Taylor series of the exact solution around 0:

\[
\frac{-\varepsilon + (4 + \varepsilon^2)^{1/2}}{2} = \frac{-\varepsilon}{2} + \left( 1 + \frac{\varepsilon^2}{4} \right)^{1/2} = \frac{-\varepsilon}{2} + 1 + \frac{\varepsilon^2}{8} \ldots
\]

and similarly for \( r^+(\varepsilon) \).

\[ \text{Fig. 2.1: Convergence of perturbation series for (R1)} \]
In this case, the perturbation series are quite accurate even for fairly large values of $\varepsilon$ (Fig. 2.1). If only the $O(\varepsilon)$ term is kept in the series for either root, the $O(\varepsilon^2)$ term gives an error estimate of roughly $\varepsilon^2/8$. If the $O(\varepsilon^2)$ term is kept in the perturbation series, it remains highly accurate out to $|\varepsilon| \approx 1.5$.

A trickier root-finding example

We wish to find perturbation series for the roots of the cubic equation

$$p(r; \varepsilon) = r^3 + r^2 - \varepsilon = 0, \quad \varepsilon \ll 1. \tag{2.3}$$

Its real roots (but not any complex roots) can be visualized by plotting $p(r; \varepsilon)$ as in Fig. 2.2; the roots are where it crosses the $r$-axis. From the graph, we can see that the unperturbed problem has a root $r_0 = -1$ and a double root $r_{0}^{II,III} = 0$ (we can tell it is a double root because $dp/dr$ is also zero at $r = 0$). The double root of the unperturbed problem is what makes the perturbation series more tricky. Graphically, we can see that for a small positive $\varepsilon$, $p(r; \varepsilon)$ locally looks like an upward facing parabola around its minimum value $-\varepsilon$ at $r = 0$. Thus we can anticipate that the two roots $r_{II,III}^{II,III}(\varepsilon)$ will be roughly $\pm O(\varepsilon^{1/2})$, rather than being linear functions of $\varepsilon$. On the other hand, the simple root $r^I(\varepsilon)$ will be $-1+O(\varepsilon)$. Let’s explore how this emerges from the perturbation series approach!

Fig. 2.2: Graph of $p(r; \varepsilon)$; roots marked as dots for $\varepsilon = 0$ and 0.1.