Cancel out the dominant balance $x^2S_0^2 + x^2 = 0$ in (17.5):

$$x^2(S''_0 + S'_r + 2S'_0S'_r + S''_r) + x(S'_0 + S'_r) - v^2 = 0$$

To find a dominant balance, first remove terms that are obviously unimportant. Since $S'_0 = \pm i$, we have $S''_0 = 0$ and $v^2 < xS'_r = \pm ix$. Since $S'_r << S'_0$, we expect $S''_r \ll S'_0S'_r$. Thus

$$x^2(S''_r + 2S'_0S'_r) + xS'_0 = 0$$

Lastly, assuming $S'_r$ is $O(x^{-a})$, then $S''_r = O(x^{-(a+1)}) \ll S'_0S'_r = \pm iS'_r = O(x^{-a})$. Thus we can neglect the first term in the above equation to obtain

$$x^2(\pm 2iS'_r) \pm ix = 0 \implies S'_r \approx S'_r = -\frac{1}{2x}$$  \hspace{1cm} (18.1)

4. **Computation of $S'_2$**

Take $S(x) = S_0(x) + S_1(x) + S_r(x)$, where $S_r(x)$ is a new remainder term. Substitute into (17.3):

$$x^2(S''_r + S'_r + S''_r + S'_r^2 + S'_0S'_r + 2S'_0S'_r + 2S'_0S'_r + 2S'_0S'_r + S'_0S'_r + x(S'_0 + S'_r + S'_r) + x^2 - v^2 = 0$$

Cancel out the lower-order dominant balances $x^2S'_0^2 + x^2 = 0$ and $x^2(2S'_0S'_r) + xS'_0 = 0$, and note $S''_0 = 0$ to get:

$$x^2(S''_r + S'_r + S'_r^2 + S'_0S'_r + 2S'_0S'_r + 2S'_0S'_r + x(S'_0 + S'_r) - v^2 = 0$$

Neglect terms which cannot be dominant - $S''_r \ll S'_r^2$, $2S'_0S'_r \ll 2S'_0S'_r$, $S'_0 \ll S'_r$, $S'_r \ll S'_1$:

$$x^2(S''_r + S'_r^2 + 2S'_0S'_r) + xS'_r - v^2 = 0$$

Substitute $S'_0 = \pm i$ and $S'_r = -1/2x$:

$$x^2\left(\frac{1}{2x^2} + \frac{1}{4x^2} \pm 2iS'_r\right) + x\left(-\frac{1}{2x}\right) - v^2 = 0$$

All the known terms are $O(1)$, so none can be neglected. We solve this equation for an asymptotic approximation $S'_2$ to $S'_r$:

$$\pm 2i\frac{3}{2} = v^2 - \frac{1}{4} \implies S'_2 = \pm \frac{i}{2x^2} \left\{ \frac{1}{4} - v^2 \right\}$$  \hspace{1cm} (18.2)

We could clearly continue this process to develop an asymptotic series in which $S'_n = O(x^{-n})$.

Putting the terms we have together,

$$S^{{r\times}}(x) = \pm i - \frac{1}{2x} \pm \frac{i}{2x^2} \left\{ \frac{1}{4} - v^2 \right\} \ldots$$
Integrating (ignoring integration constants which are additive constants in $S(x)$ and hence multiplicative constants in $e^S$, and so do not lead to new solutions) we find:

$$S^\pm(x) = \pm ix - \frac{1}{2} \log x \pm i \frac{v^2 - \frac{1}{4}}{2x} \ldots$$

$$y^\pm(x) = \exp[S^\pm(x)] = \exp\left[ \pm ix - \frac{1}{2} \log x \pm i \frac{v^2 - \frac{1}{4}}{2x} \ldots \right] = x^{-1/2} \exp\left[ \pm i x + i \frac{v^2 - \frac{1}{4}}{2x} \ldots \right]$$

These solutions are (to within constants) asymptotic series for the Hankel functions of order $n$:

$$H^{(1,2)}_\nu(x) = J_\nu(x) \pm i Y_\nu(x) = \left( \frac{2}{\pi} \right)^{1/2} e^{\mp i (\nu \pi/2 + \pi/4)} y^\pm(x) \sim \left( \frac{2}{\pi x} \right)^{1/2} e^\pm i (x - \nu \pi/2 - \pi/4 + (\nu^2 - 1/4)/2x \ldots)$$

We can get corresponding series for $J_\nu(x)$ and $Y_\nu(x)$ by taking real and imaginary parts of the above.

The plot below, made by Matlab script `bessel_large_x.m` (see class web page) shows the $\exp(S_0 + S_1)$ approximations to $J_0(x)$ and $Y_0(x)$ are amazingly accurate down to $x = 2.5$, and adding the $S_2$ correction term makes them accurate down to $x = 1.5$. Thus one could use the first three terms of the Frobenius series for $|x| < 1.5$ and the first three terms of the large-$x$ asymptotic series for $|x| > 1.5$ to get an excellent, computationally efficient global approximation to $J_0(x)$ and $Y_0(x)$. 

![Asymptotic series approximations exp(S0+S1) to J0(x) (solid) and Y0(x) (dashed)](attachment:image)