Approximate solution by dominant balance near an irregular singular point (B&O 3.4-3.5)

Near an irregular singular point $x_0$ of a linear homogeneous ODE, we can find an asymptotic series solution via the method of dominant balance. Most common ODEs have irregular singular points at $x_0 = \infty$, so this is a very powerful method for finding the asymptotic behavior of solutions of an ODE for large $x$. We will demonstrate the method on a 2nd order ODE

$$y'' + p_1(x)y' + p_0(x)y = 0,$$

(17.1)

but it can be applied to ODEs of other orders as well. It works as follows:

1. As with WKB, set $y(x) = e^{S(x)}$, and derive a nonlinear ODE for $S'$,

$$S'' + S'' + p_1(x)S' + p_0(x) = 0$$

(17.2)

2. We look for a consistent dominant balance on the LHS of (17.2) as $x \to x_0$, from which we deduce a leading-order asymptotic behavior $S_0(x)$ of the solutions. Two key facts are

   a. Near an irregular singular point, we expect $y(x)$ to be more singular than at a regular singular point (since the singularity in the coefficients is stronger). Thus $y(x)$ should grow or decay faster than any power of $x - x_0$, so $S(x)$ should have a singularity stronger than $O[\log(x - x_0)]$. This is consistent with the behavior $S(x) = O[(x - x_0)^r]$, $r > 0$, whence $S' = O[(x - x_0)^{r+1}]$ and $S'' = O[(x - x_0)^{r+2}]$. Thus $S''/S^2 = O[(x - x_0)^{r-2}/(x - x_0)^{-2r-2}] = O[(x - x_0)^r] \to 0$ as $x \to x_0$.

   Hence we can assume $S'' \ll S'^2$. This ensures that the leading order balance in (17.2) is a quadratic equation in $S'$, which is sure to be solvable.

   b. Include only the leading asymptotic behavior of $p_1(x)$ and $p_0(x)$ to get the simplest balance.

3. Now write $S' = S'_0 + S'_1$, substitute into (17.2), and look for a new dominant balance for the residual $S'_1$, whose asymptotic solution we call $S_1(x)$. In constructing the dominant balance note that $S'_1/S'_0 \to 0$ as $x \to x_0$. Assuming that $S'_1$ and $S'_0$ go like powers of $x - x_0$, this also implies $S''_1/S'_0 \to 0$.

4. Repeat step (3) until satisfied or exhausted to get an asymptotic series

$$S(x) = S_0(x) + S_1(x) + S_2(x) \ldots$$

as $x \to x_0$.
Dominant balance on Bessel’s equation as \( x \to \infty \)

It is easily shown that \( x_0 = \infty \) is an irregular singular point of Bessel’s equation of order \( \nu \). We follow the method just described to generate asymptotic formulas for its solutions for large \( x \).

1. Set \( y = e^{S(x)} \). Then Bessel’s equation can be written
   \[
   x^2 (S'' + S'^2) + xS' + x^2 - \nu^2 = 0
   \]  
   (17.3)

2. Look for a dominant balance for large \( x \). As discussed earlier, we neglect \( S'' \ll S'^2 \). In addition, \( \nu^2 \ll x^2 \). Thus we have
   \[
   \frac{x^2 S'^2}{A} + \frac{xS'}{B} + \frac{x^2}{C} \approx 0 \quad \text{for} \quad |x| \gg 1
   \]  
   (17.4)

We could solve this quadratic for \( S' \), and look for its dominant behavior at large \( x \). However, it is easier to find a consistent dominant balance in (17.4), then solve for \( S' \). The three choices are:

(i) \( |A| \approx |B| >> |C| \)
   \[
   x^2 S'^2 + xS' \approx 0 \quad \Rightarrow \quad S' = -x^{-1} \text{ or } 0
   \]

   Neither solution is consistent since \( |C| = x^2 >> |A|, |B| = O(1) \) or 0.

(ii) \( |B| \approx |C| >> |A| \)
   \[
   xS' + x^2 \approx 0 \quad \Rightarrow \quad S' = -x
   \]

   This balance is inconsistent since \( |A| = |x^2 S'^2| = x^4 >> |B|, |C| = O(x^2) \).

(iii) \( |A| = |C| >> |B| \)
   \[
   x^2 S'^2 + x^2 \approx 0 \quad \Rightarrow \quad S' = \pm i
   \]

   This balance is consistent since \( |B| = |xS'| = O(x) << |A|, |C| = O(x^2) \).

Thus the leading-order asymptotic approximation for \( S' \) for large \( x \) is

\[
S_0'(x) = \pm i
\]

The two choices correspond to two linearly independent solutions of Bessel’s equation.

3. Set \( S' = S'_0 + S'_r \) and find a dominant balance for the residual \( S'_r \). Since \( S'_0 \) is the leading-order asymptotic solution, \( S'_r \ll S'_0 \) for large \( |x| \).

Substitute into (17.3):

\[
\frac{x^2}{A} (S'' + S'^2) + \frac{x}{B} (S' + S^2) + x(S'_0 + S'_r) + x^2 - \nu^2 = 0
\]  
   (17.5)