The asymptotic form of the two solutions as \( x \to 0 \) is \( y_{1,2}(x) \sim x^{\nu - \nu} \) unless \( \nu = 0 \), when \( y_1 \sim 1 \) and \( y_2 \sim \log(x) \). As is often true, the most relevant cases correspond to complications (\( \nu = 0 \leftrightarrow \) double root; \( \nu = n \leftrightarrow \alpha_1 - \alpha_2 = 2n = \text{integer} \)). Regardless, one solution will always have the form 
\( y_i(x) = y(x; \nu) \), where

\[
y(x; \alpha) = x^\alpha \sum_{n=0}^{\infty} a_n(\alpha) x^n = \sum_{n=0}^{\infty} a_n(\alpha)x^{\nu + \alpha}, \tag{15.2}
\]

We write \( y(x; \alpha) \) as a function of \( \alpha \) rather than immediately setting \( \alpha = \nu \) so that

(a) If \( \alpha_1 - \alpha_2 = 2\nu \) is not an integer, we can derive \( y_2(x) = y(x; -\nu) \).

(b) For the case of equal roots \( \nu = 0 \), we can find \( y_2(x) = \partial y(x; \alpha) / \partial \alpha |_{\alpha = \nu} \).

Now (15.2) implies

\[
y''(x) = \sum_{n=0}^{\infty} (n+\alpha)(n+\alpha-1)a_n(x^{\nu + \alpha - 2})
\]

\[
y'(x) = \sum_{n=0}^{\infty} (n+\alpha)a_n(x^{\nu + \alpha - 2})
\]

\[
\left(1 - \frac{\nu^2}{x^2}\right)y = \sum_{m=0}^{\infty} a_m(x^{\nu + \alpha} - \nu^2 \sum_{n=0}^{\infty} a_n(x^{\nu + \alpha - 2}) = \sum_{m=n+2}^{\infty} a_{n+2}(x^{\nu + \alpha - 2} - \nu^2 \sum_{n=0}^{\infty} a_n(x^{\nu + \alpha - 2})
\]

Substituting into (BE) and collecting like powers of \( x \),

\( n = 0 \) (\( x^{\alpha - 2} \)): \( \alpha(\alpha - 1) + \alpha - \nu^2 = 0 \) (the indicial eqn, automatically satisfied by \( \alpha = \nu \).)

\( n = 1 \) (\( x^{\alpha - 1} \)): \( [(\alpha + 1)\alpha + \alpha + 1 - \nu^2]a_1 = 0 \Rightarrow a_1 = 0 \)

\( n \geq 2 \) (\( x^{\alpha + n - 2} \)): \( [(\alpha + n)(\alpha + n - 1) + \alpha + n - \nu^2]a_n = -a_{n-2} \Rightarrow a_n = -\frac{a_{n-2}}{(\alpha + n - \nu)(\alpha + n + \nu)} \)

Note that an inconsistency is possible if \( \alpha + n = \nu \), because the coefficient on the left hand side will be zero, and the right hand side has been determined in a previous iteration and may not be zero. This can’t happen for the positive root \( \alpha_1 = \nu \), but for \( \alpha_2 = -\nu \), inconsistency can occur if \( n = 2\nu = \alpha_1 - \alpha_2 \). This is why we need to make an exception for the case that \( \alpha_1 - \alpha_2 \) is an integer.

Iterating, we obtain:

\[
a_{2p}(\alpha) = \frac{(-1)^p a_0(\alpha)}{(\alpha + 2p - \nu)(\alpha + 2p - 2 - \nu)(\alpha + 2 - \nu)(\alpha + 2 + \nu)(\alpha + 2p + \nu)(\alpha + 2p - 2 + \nu)}
\]
If $a_{2p+1}(\alpha) = 0$

At this point, we take $\alpha = \nu$ (the ‘true’ $\alpha$), whence

$$a_{2p}(\nu) = \frac{(-1)^p a_0(\nu)}{(2p)(2p-2)\ldots(2)(2p+2)(2p-2+2\nu)\ldots(2+2\nu)} = \frac{(-1)^p a_0(\nu)}{2^m p!(p+\nu)\ldots(1+\nu)}$$

Making use of the gamma function $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$, for which $\Gamma(n+1) = n!$ and $\Gamma(z+1) = z\Gamma(z)$, we can write $(p+\nu)\ldots(1+\nu) = \Gamma(\nu+p+1)/\Gamma(\nu+1)$. Hence

$$y_1(x) = a_0 \Gamma(\nu+1)2^\nu \sum_{p=0}^\infty \frac{(-1)^p (x/2)^{2p+\nu}}{p!\Gamma(\nu+p+1)}$$

With appropriate normalization $a_0(\nu) = 2^{-\nu}/\Gamma(\nu+1)$, this is the Frobenius series for $J_\nu(x)$, the Bessel function of order $\nu$.

**The second ($\alpha_2 = -\nu$) solution**

1. If $\alpha_1 - \alpha_2 = 2\nu$ is noninteger, the second solution is $y_2(x) = y(x; -\nu) = J_\nu(x)$.
2. If $2\nu = 2P + 1$ is odd, this solution still works because the $n = 2P + 1$ terms in the Frobenius series give $[(\nu + 2P + 1)^2 - \nu^2]a_{2P+1} = -a_{2P-1} = 0$. Although the LHS is zero, which could have produced an inconsistency, the RHS is also zero since the odd coefficients are zero. Thus there is no inconsistency in taking $a_{2P+1} = 0$.
3. If $2\nu = 2P$ is even and positive, the second solution is too messy for in-class derivation.
4. If $\nu = 0$, the second solution is

$$y_2(x) = \frac{\partial}{\partial \alpha} y(x; \alpha) \bigg|_{\alpha=0} = \sum_{p=0}^\infty \frac{\partial}{\partial \alpha} \left[a_{2p}(\alpha)x^{2p+\alpha}\right]_{\alpha=0}$$

$$= \sum_{p=0}^\infty \left[a_{2p}(\alpha)x^{2p+\alpha}\log x + \frac{\partial a_{2p}(\alpha)}{\partial \alpha} x^{2p+\alpha}\right]_{\alpha=0} = J_0(x)\log x + \sum_{p=0}^\infty b_{2p} x^{2p}$$

where

$$b_{2p} = \frac{\partial a_{2p}}{\partial \alpha} \bigg|_{\alpha=0} = a_{2p} \frac{\partial}{\partial \alpha} \log a_{2p}(\alpha) \bigg|_{\alpha=0}$$

$$= a_{2p}(0) \frac{\partial}{\partial \alpha} \left[-2\sum_{k=0}^p \log(\alpha + 2k)\right]_{\alpha=0} = a_{2p}(0) \left[-2\sum_{k=0}^p \frac{1}{\alpha + 2k}\right]_{\alpha=0} = \frac{(-1)^{p+1}}{2^{2p}(p!)^2} H_p$$
and \( H_p = \sum_{k=1}^{p} \frac{1}{k} \) is the \( p \)'th harmonic number. The Bessel function of the second kind of order zero is \( Y_0(x) = \frac{2}{\pi} \left\{ y_2(x) + (\gamma - \log 2) J_0(x) \right\} \), a linear combination of the two Frobenius solutions.

Here \( \gamma = \lim_{n \to \infty} \left\{ H_n - \log n \right\} = 0.577 \ldots \) is Euler’s constant.

The functions \( J_0(x) \) and \( Y_0(x) \) are the functional analogues to \( \cos x \) and \( \sin x \) in cylindrical coordinates. They can be combined into the Hankel functions of order zero, \( H_0^{(1)}(x) = J_0(x) + i Y_0(x) \) and \( H_0^{(2)}(x) = J_0(x) - i Y_0(x) \), which function like \( \exp(\pm ix) \) and are useful for solving such problems as finding the structure of waves radiated from a long linear antenna.

The plot below, generated by Matlab script `bessel_plot.m` (class web page) shows the convergence of the Frobenius series for \( J_0(x) \) and \( Y_0(x) \) truncated after the \( x^{2P} \) term for various \( P \). With \( P = 13 \), both series are quite accurate for \( 0 < x < 10 \).