Example: Refraction of an oblique light wave by a change in refractive index \( N(z) \)

Consider a medium unbounded in \( x \) and \( z \) whose refractive index is constant for \( z < z_1 \), then smoothly increases from \( N_1 \) to \( N_2 \) in \( z_1 < z < z_2 = z_1 + H \), and is constant for \( z > z_2 \). In this example, we use WKB (dimensional form) to calculate what happens to a short wavelength wave impinging on this refractive index gradient, as in the sketch to the right.

The electrical potential \( u(x, z, t) \) obeys the 2D wave equation

\[
u_{xx} + u_{zz} - \frac{N^2(z)}{c_0^2} u_{tt} = 0 \tag{11.1}\]

We consider a planar monochromatic light wave \( u(x, z, t) = A \exp(ikx + im_1z - i\omega t) \) in the region \( z < z_1 \) propagating obliquely up and right toward the refractive index gradient in the direction \( \mathbf{k} = ki + m_1j \) of its vector wavenumber, which has an incidence angle \( \theta_1 = \cot^{-1}(m_1/k) \) away from the vertical.

To satisfy (11.1) for \( z < z_1 \), the wave frequency \( \omega \) must obey the dispersion relation

\[
k^2 + m_1^2 = \frac{N_1^2 \omega^2}{c^2} \tag{11.2}\]

What happens to the wave when it hits the gradient region? A solution to (11.1) of this structure can be sought in the form \( u(x, z, t) = e^{i(kx-\omega t)}y(z) \):

\[
0 = \frac{d^2y}{dz^2} + \left( \frac{N^2(z)\omega^2}{c_0^2} - k^2 \right) y, \quad \text{where } y(z) \sim A \exp(im_1z) \text{ as } z \to -\infty \tag{11.3}\]

Using the dispersion relation (11.2), we can write

\[
m^2(z) = \frac{N^2(z)\omega^2}{c_0^2} - k^2 = \frac{N_1^2(z)}{N_1^2} \left( k^2 + m_1^2 \right) - k^2 = k^2 \left( \frac{N_1^2(z)}{N_1^2 \sin^2 \theta_1} - 1 \right). \tag{11.4}\]

When \( kH \ll 1 \), the wavelength will be short compared to the length scale of the refractive index change, so we can apply the formula (10.6b) (except with \( k, x \) replaced by \( m_1, z \)) to obtain WKB approximations to the two linearly independent solutions:
\[ y^\pm(z) = |m(z)|^{-1/2} \exp \left\{ \pm i \int_{z_0}^z m(\zeta) d\zeta \right\} \left\{ 1 + O(m'/m^2) \right\} \]

Note that since \( N^2(z) \) is \( O(1) \) and varies over a distance \( H \), (11.4) implies that \( m = O(k) \) and \( m' = O(k/H) \), so \( m'/m^2 = O(1/kH) \).

To match the form of the incident wave as \( z \to -\infty \), we take \( A \) times the positive solution:

\[ y(z) = A|m(z)|^{-1/2} \exp \left\{ \int_{z_0}^z m(\zeta) d\zeta \right\} \left\{ 1 + O(1/kH) \right\}, \quad kH \gg 1 \quad (11.5) \]

which can be redimensionalized to the form

\[ u(x,z,t) = A|m(z)|^{-1/2} \exp \left\{ kx + \int_{z_0}^z m(\zeta) d\zeta - \omega t \right\} \left\{ 1 + O(\frac{1}{kH}) \right\} \quad (11.6) \]

Thus, the WKB asymptotic solution is a sinusoidal wave whose vertical wavenumber \( m(z) \) changes as it moves across the refractive index gradient. This bends the direction of wave propagation (wave refraction). According to (11.4), \( m \) increases as \( N \) increases; in fact from (11.4) it is easy to show Snell’s law that the angle of incidence of the wave obeys \( N(z) \sin \theta = N_1 \sin \theta_1 \).

Fig. 11.3 shows \( \text{Re}(u) \) for a specific example

\[ N(z) = N_1 + (N_2 - N_1) \left\{ \text{erf}(z) + 1 \right\} / 2, \quad N_1 = 1, \quad N_2 = 2 \quad (11.7) \]

in which \( N \) changes over roughly over a distance \( H = 2 \) between \( z_1 = -1 \) and \( z_2 = 1 \) with an incident wave with \( k = m_1 = 2\pi \) so \( kH = 4\pi \gg 1 \), so WKB is quite accurate. Note the refraction of the wave toward the vertical as well as the amplitude reduction where \( N \) is larger.

If \( N \) increases with \( z \) as in this example, the WKB approximate solution consists purely of an upward-propagating wave. That is, there is no wave reflection in the WKB asymptotic limit that the wavelength is much shorter than the length scale \( H \) of the refractive index change. This can be contrasted to a step increase in refractive index (i.e. over a distance much shorter than the wavelength), which can be shown by direct solution of (11.3) to create a reflected wave of amplitude \((N_2 - N_1) / (N_2 + N_1) = 1/3\) in this case, as well as a transmitted refracted wave. In general, partial reflection not predicted by the WKB approximation will occur wherever the medium varies over length scales shorter than a wavelength of the wave.
Fig. 11.3: WKB solution for wave refraction across a smooth increase in refractive index from 1 to 2 across the region between the blue dashed lines.