WKB Theory: Waves in Slowly-Varying Media

Another important class of singular perturbation problems has the form

\[ y'' - \varepsilon^2 Q(x)y = 0, \quad 0 < \varepsilon \ll 1. \]  

(10.1)

where \( Q(x) \) is some \( O(1) \) smoothly varying function of a length coordinate \( x \) that has been nondimensionalized such that \( O(1) \) variations of \( Q(x) \) occur over \( O(1) \) distances in \( x \). The asymptotic approximation for its solution is called WKB (Wentzel-Kramers-Brillouin) theory. This class of equation commonly occurs in the study of linear waves propagating through spatially inhomogeneous media; we will study one such example in the next lecture. In that case, the small parameter \( \varepsilon \) will measure the ratio of the wavelength to the typical inhomogeneity lengthscale.

While (10.1) is certainly a singular perturbation problem (because the second order ODE reduces to the zeroth order ODE \( y = 0 \) if \( \varepsilon = 0 \)) it is not immediately obvious how to start an asymptotic approximation to the solution in the limit of small \( \varepsilon \). There are only two terms in (10.1) so it cannot be further simplified using a dominant balance between these terms. Instead, we recognize that we can asymptotically solve (10.1) locally, within a distance \( O(\varepsilon) \) of some arbitrary reference point \( x_0 \).

In this range, \( Q(x) = Q(x_0) + O(\varepsilon) \), and the approximate asymptotic general solution of the ODE has the form \( y(x) \sim c_1 \exp \{Q^{1/2}(x_0)x/\varepsilon\} + c_2 \exp \{-Q^{1/2}(x_0)x/\varepsilon\} \). Where \( Q(x_0)>0 \), this gives rapid exponential growth/decay; \( Q(x_0)<0 \) gives rapid sinusoidal oscillations. The trick is how to generate a solution valid for all \( x \) which locally has these characteristics.

To do this, we focus on the function inside the exponent, which we saw above locally varies linearly with a slope of \( O(\varepsilon^{-1}) \), by setting

\[ y(x) = \exp(S(x)) \]  

(10.3)

Substituting into (10.1) yields

\[ y'' = (S'e^S)' = (S'' + S'^2)e^S \Rightarrow \frac{r'}{A} + \frac{r^2}{B} = \frac{Q(x)}{\varepsilon^2} \]  

(10.4)

We’ve replaced a 2\(^{nd}\)-order linear ODE with a 1\(^{st}\)-order nonlinear ODE (not necessarily a good trade!), but now we can do dominant balance on \( r \). Our local argument implied that for \( x \) close to any reference point \( x_0 \), \( r(x) \approx \pm Q(x_0)^{1/2}/\varepsilon \). Thus we try the dominant balance \( B \sim C >> A \), which gives the Eikonal equation

\[ r^2 \sim Q(x)/\varepsilon^2 \Rightarrow S' = r \sim r_0(x)/\varepsilon, \quad r_0 = \pm Q^{1/2}(x) \]  

(10.5)
This dominant balance is consistent as long as \( r'_0 \ll r_0^2 \Rightarrow \varepsilon \left| Q' / Q^{3/2} \right| \ll 1 \). As long as \( Q(x) \) is a smooth function of \( x \), this is true except at turning points \( x_t \) where \( Q(x_t) = 0 \). The asymptotic behavior around a turning point is determined by approximating \( Q(x) \approx (x - x_t) Q'(x_t) \), for which (10.1) turns into \( y'' - \varepsilon^2 (x - x_t) Q'(x_t) y = 0 \), a scaled version of Airy’s equation which can be solved in terms of special functions called Airy functions that are related to Bessel functions. Using the asymptotic behavior of the Airy function for large arguments, we will later derive useful connection formulas connecting asymptotic solutions to (10.1) for \( x > x_t \) with solutions valid for \( x < x_t \).

We can develop an asymptotic series \( r(x) \sim \varepsilon^{-1} (r_0 + \varepsilon^q r_1 + \varepsilon^{2q} r_2 \ldots) \) and substitute into (10.4):

\[
\varepsilon^{-1} (r_0 + \varepsilon^q r_1 + \varepsilon^{2q} r_2 \ldots) + \varepsilon^{-2} (r_0 + \varepsilon^q r_1 + \varepsilon^{2q} r_2 \ldots)^2 = \varepsilon^{-2} Q(x)
\]

The \( O(\varepsilon^2) \) terms give the leading order solution (10.5). To balance the first neglected term \( A \), which is \( O(\varepsilon^{-1}) \), we have to choose \( q = 1 \). Then, the next order in \( \varepsilon \) gives

\[
O(\varepsilon^{-1}): \quad r'_0 + 2 r_0 r_1 = 0 \Rightarrow r_1 = -r'_0 / 2 r_0 = -Q' / 4 Q
\]

Putting the series back together

\[
S' = \pm \varepsilon^{-1} Q^{1/2} (x) - Q' / 4 Q + O(\varepsilon)
\]

\[
\Rightarrow \quad S^\pm (x) = \pm \varepsilon^{-1} \int_{x_0}^x Q^{1/2} (\zeta) d\zeta - \frac{1}{4} \log |Q(x)| + O(\varepsilon)
\]

\[
\Rightarrow \quad y^\pm (x) = |Q(x)|^{-1/4} \exp \left\{ \pm \varepsilon^{-1} \int_{x_0}^x Q^{1/2} (\zeta) d\zeta \right\} \left[ 1 + O(\varepsilon) \right]
\]

This WKB approximation to the two linearly independent solutions of (10.1) is asymptotically accurate in the limit \( \varepsilon \to 0 \). If \( Q > 0 \), the two solutions \( y^\pm (x) \) are the exponentially growing and decaying analogues of the two local solutions at the top of the previous page, with local e-folding scale \( Q^{1/2} / \varepsilon \). If \( Q < 0 \), the two solutions can be regarded as right and left-propagating waves with spatially-varying local wavenumber \( |Q|^{1/2} / \varepsilon \) and local amplitude \( |Q|^{-1/2} \).

Application of WKB to dimensional problems with no explicit small parameter \( \varepsilon \)

Commonly, the WKB approximation is applied to a **dimensional** ODE

\[
y'' - q(x) y = 0
\]
of the form (10.1) with **no explicit small parameter**. Formally, the dominant balance approach starting with \( y(x) = \exp\{S(x)\} \) will give the same solutions (but now with \( \varepsilon \) replaced by 1):

\[
y^\pm(x) = \left|q(x)\right|^{-1/4} \exp\left\{ \pm \int_{x_0}^{x} q^{1/2}(\zeta) d\zeta \right\} \left\{ 1 + O(\left|\varepsilon/\varepsilon\right|^{3/2}) \right\} \quad \text{if} \quad \left|\varepsilon/\varepsilon\right|^{3/2} \ll 1 \quad (10.6a)
\]

Here, the relative error estimate is based on the ratio between successive terms in the asymptotic series for \( S' \). A common alternative form of (10.1a), especially in wave propagation problems, is to set \( q(x) = -k^2(x) \).

\[
y'' + k^2(x)y = 0 \quad (10.1b)
\]

We interpret \( k(x) \) as a local wavenumber for oscillations. In this notation, the two WKB solutions (10.6a) are:

\[
y^\pm(x) = \left|k(x)\right|^{-1/2} \exp\left\{ \pm i \int_{x_0}^{x} k(\zeta) d\zeta \right\} \left\{ 1 + O(\left|\varepsilon/\varepsilon\right|^{3/2}) \right\} \quad \text{if} \quad \left|\varepsilon/\varepsilon\right|^{3/2} \ll 1 \quad (10.6b)
\]

These solutions are still formally valid if \( k^2(x) < 0 \), in which case \( k \) is imaginary and we get decaying and growing exponentials. We can visualize the condition for validity of WKB as a ‘slowly varying medium approximation’ that the relative change of \( k \) in one wavelength \( \lambda = 2\pi/k \) is small, i.e. that

\[
\frac{\Delta k}{k} = \frac{\lambda k'}{k} \propto \frac{k'}{k^3} \ll 1
\]

(we have absorbed the order-1 factor \( 2\pi \) into the definition of ‘\( \ll \)’).

**Sketch of WKB solutions**

Sketches of the solutions are shown in Fig. 11.1. If \( Q > 0 \), the two solutions \( y^\pm(x) \) are the exponentially growing and decaying analogues of the two local solutions at the top of page 10.2, with local e-folding scale \( \varepsilon/Q^{1/2} \). If \( Q < 0 \), the two solutions are right and left-propagating waves with spatially-varying local wavenumber \( k = |Q|^{1/2}/\varepsilon \) and local amplitude \( |Q|^{1/4} \propto |k|^{-1/2} \).

![Fig. 11.1: Sketches of WKB solutions](image-url)