3.1 Basic Equations in isobaric coordinates (x,y,p,t)

The horizontal momentum equation is

\[ \frac{D\mathbf{V}}{Dt} + f\hat{k} \times \mathbf{V} = -\nabla_p \phi \]

where \( \mathbf{V} = u\hat{i} + v\hat{j} \) is the horizontal velocity and the little subscript p means holding pressure constant. The total derivative written out in component form is coordinate system dependent. Even though this is only the horizontal momentum equation, \(\frac{D}{ Dt}\) still depends on vertical advection:

\[ \frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + \omega \frac{\partial}{\partial p} \]

where \( \omega = \frac{Dp}{Dt} \) is the vertical velocity component in isobaric coordinates. \( \omega \) has the opposite sign of \( w \).

The geostrophic wind definition has no \( \rho \)

\[ f\mathbf{V}_g = \hat{k} \times \nabla_p \phi \]

For homework this week you get to show an additional extremely nice property of \( \mathbf{V}_g \) in isobaric coordinates — it is divergentless on pressure surfaces when \( f=\text{constant} \):

\[ \nabla_p \cdot \mathbf{V}_g = 0 \]

The continuity equation also has no \( \rho \)

\[ \nabla_p \cdot \mathbf{V} + \frac{\partial \omega}{\partial z} = 0 \]

The thermodynamics energy equation in isobaric coordinates is

\[ \frac{\partial T}{\partial t} + \mathbf{V} \cdot \nabla_p T - S_p \omega = J/c_p \]

where the static stability in isobaric coordinates is

\[ S_p = -\frac{T}{\theta} \frac{\partial \theta}{\partial p} = \frac{\alpha}{c_p} - \frac{\partial T}{\partial p} = \frac{\Gamma_d - \Gamma}{g_p}. \]
Compare with the z-coordinate equation

\[ S = \frac{T \partial \theta}{\theta \partial z} = \Gamma_d - \Gamma. \]

For reasonably small vertical displacements we can usually approximate \( \Gamma \approx \) constant, hence \( S \approx \) constant. However \( S_p \) is not approximately constant because \( \rho \) varies exponentially. This is a disadvantage of isobaric coordinates.

### 3.2 Balanced Flow

Balanced flows follow relatively simple force balances.

Here we let \( \omega = 0 \) and only consider flow that are approximately in the horizontal plane. \( \omega \) resurfaces again near the end of this chapter.

#### 3.2.1 Natural Coordinates

\( \hat{i} = \) tangent to velocity at each instant

\( \hat{n} = \) normal to velocity at each instant

\( s \) is the distance along the parcel’s path

\( \mathbf{V} = V \mathbf{\hat{i}} \) is the velocity

\( V = Ds/Dt \) is the speed

\[ \frac{DV}{Dt} = \mathbf{i} \frac{DV}{Dt} + V \frac{D\hat{i}}{Dt} \]

\[ \frac{D\hat{i}}{Dt} = \lim_{\delta t \to 0} \frac{\delta \hat{i}}{\delta t} \]

\[ \delta \hat{i} = \delta \psi \mathbf{\hat{n}} \]

\[ \delta \psi = \frac{\delta s}{R} \]

\[ \delta s = V \delta t \]

\[ \frac{D\hat{i}}{Dt} = \frac{V}{R} \mathbf{\hat{n}} \]

Fig 3.1 from Holton illustrates the equations on the left
replace p 3 of Wed.’s notes with this page

\[
\frac{D\mathbf{V}}{Dt} = \mathbf{i} \frac{DV}{Dt} + \mathbf{n} \frac{V^2}{R}
\]  

Eq. (1)

The Coriolis force in natural coordinates is

\[-f \mathbf{k} \times \mathbf{V} = -fV\mathbf{n}.\]

The pressure gradient force (PGF) in natural coordinates is

\[-\nabla \Phi = -\mathbf{i} \frac{\partial \Phi}{\partial s} - \mathbf{n} \frac{\partial \Phi}{\partial n}\]

Setting the acceleration (Eq. 1) equal to the sum of Coriolis and PGF gives two component equations (no vector symbols now in component equations)

\[
\frac{DV}{Dt} = -\frac{\partial \Phi}{\partial s} \mathbf{i} \text{--component (2)}
\]

\[
\frac{V^2}{R} + fV = -\frac{\partial \Phi}{\partial n} \mathbf{n} \text{--component (3)}
\]

Eq 2 states the parcel acceleration along the parcel path equals the PGF along the parcel path. Eq 3 states the acceleration normal to the parcel path (which is the centrifugal force) plus the Coriolis acceleration equals the PGF normal to the parcel path.

By definition, \( V > 0 \) but \( R \) can be positive or negative: \( R > 0 \) for cyclonic flow and \( R < 0 \) for anticyclonic or antiharic (clockwise flow around a low) flow
3.2.2 Geostrophic Flow

IF $|R| \to \infty$ then $V \to V_g$. No matter what $R$ is, so long as $DV/DT = 0$ then

$$fV_g = -\frac{\partial \Phi}{\partial n}$$

where the geostrophic wind is the velocity we find from setting Coriolis force and PGF equal. $V_g$ is locally parallel to height contours, and because $DV/DT = 0$, $V_g$ and $V$ are either parallel or antiparallel. If $DV/Dt \neq 0$ then $f\hat{k} \times V_g = -\nabla \Phi$ instead, and you can expect a nonzero angle between $V_g$ and $V$.

3.2.3 Inertial Flow

When $\partial \phi/\partial n = 0$ the flow is called “inertial” and the remaining balance of centrifugal and Coriolis forces yields a circular flow with $R = -V/f$. $R < 0$ always for inertial flow, so the motion is clockwise in the northern hemisphere.

3.2.4 Cyclostrophic Flow

Horizontal scales are small enough to neglect the Coriolis force:

$$\frac{V^2}{R} = -\frac{\partial \Phi}{\partial n}.$$  

Hence

$$V = \left(-R\frac{\partial \Phi}{\partial n}\right)^{1/2}.$$  

$C_e$ always points away from the center of rotation, so the PGF must always point towards it. Remember $V$ is always positive, but nothing requires the flow to be clockwise or counterclockwise. Hence it can be either. The “normal” direction is always to the left of the direction of flow.
3.2.5 Gradient flow

Gradient flow is a special case when $DV/Dt = 0$ so $V$ is time-independent and always flows parallel to lines of constant geopotential, defined by Eq 2:

$$V = -\frac{fR}{2} \pm \left(\frac{f^2 R^2}{4} - R \frac{\partial \phi}{\partial n}\right)^{1/2} = -\frac{fR}{2} \pm \left(\frac{f^2 R^2}{4} + fRV_g\right)^{1/2}$$  (4)

$V$ must be real so the argument of the square root must be positive. This is only an issue for anticyclonic flow around a high because $R < 0$ and $V_g > 0$

$$fV_g = -\frac{\partial \phi}{\partial n} < -\frac{Rf^2}{4}.$$  (5)

This requirement makes pressure gradients flat at the center of highs.
**In class exercise:** Compute the gradient wind speed

\[ V = \frac{-fR}{2} \pm \left( \frac{f^2R^2}{4} - R \frac{\partial \phi}{\partial n} \right)^{1/2} \]

for the following cases (all with \( f = 10^{-4} \) s\(^{-1}\))

1. a regular low with \(-\partial \phi/\partial n = 0.86 \times 10^{-3} \) m/s\(^2\) and \( R = 250 \) km.
2. an anomalous low with \(-\partial \phi/\partial n = -0.86 \times 10^{-3} \) m/s\(^2\) and \( R = -250 \) km.
3. a regular high with \(-\partial \phi/\partial n = 0.26 \times 10^{-3} \) m/s\(^2\) and \( R = -250 \) km.
4. an anomalous high with \(-\partial \phi/\partial n = 0.26 \times 10^{-3} \) m/s\(^2\) and \( R = -250 \) km.
5. Is Equation 4 violated for a high if \(-\partial \phi/\partial n = 0.86 \times 10^{-3} \) m/s\(^2\) and \( R = -250 \) km.
6. What will the wobbly path of the actual parcel motion look like if the low is changed to a high? (Do not make the gradient wind approximation. Just be qualitative.)