Boussinesq approximation (for ocean, atm bound layer, lab expts, ideal models) (6.6.4, but scaling not discussed)

Idea: Sometimes we can treat a compressible fluid as approximately incompressible. Formally, this is the Boussinesq approximation, which consists of:

1. Neglecting density variations \( \frac{1}{\rho} \frac{dp}{dt} \) in the mass continuity equation to get
   \[ \nabla \cdot \vec{u} = 0 \] (filters out sound waves by effectively making \( c_s = \infty \))

2. Linearizing density variations in the momentum equation:
   \[ \frac{D\bar{u}}{Dt} = -\frac{1}{\rho_0} \nabla p' + \vec{b} \]
   \[ B = -g \frac{\rho_0}{\rho_0}, \rho_0 \] is a constant reference density.
   \[ \rho_0 \] is a fluid density.

Validity: We will show the Boussinesq approximation is valid if

1. Density variations throughout the fluid are small (<20%, say), so there is a reference density \( \rho_0 \) such that \( \rho - \rho_0 \ll \rho_0 \) everywhere.

2. The generalized Mach number \( \frac{L}{c_s T} \) is small, where \( L \) is the length scale and \( T \) the timescale of the flow (\( \leq \frac{L}{U} \)) Unlike hydrostatic approx does not require \( H \ll L \), so good for turbulence.

Derivation (Spiegel and Veneris 1960 Astrophysical J., 131, 442-447)

The derivation of (2) follows the initial steps of our derivation of the hydrostatic approximation but with a constant \( \rho_0 \) that may differ from the mean fluid density at any particular height \( z \). This uses only assumption (1).

To derive (1), we must show \( \left[ \frac{1}{\rho} \frac{dp}{dt} \right] \) is much less than the individual terms \( \frac{\partial \rho}{\partial x}, \frac{\partial \rho}{\partial y}, \frac{\partial \rho}{\partial z} \) of \( \nabla \cdot \vec{u} \), so the dominant balance in the mass continuity eqn. is of those individual terms with each other.

Let \( U \) be hor. velocity scale, \( L \) and \( H \) the horizontal/sheath length scales.

Then
\[ \left[ \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial z} \right] = \frac{U}{L} \]
\[ \left[ w \right] = \frac{U}{H} \]
Now, for an adiabatic flow on which entropy \( \eta \) is conserved following fluid parcel:

\[ \frac{1}{\rho} \frac{dp}{dt} = \frac{1}{\rho} \left( \frac{\partial p}{\partial \rho} \right)_H \frac{dp}{dt} = \frac{1}{\rho c_s^2} \frac{dp}{dt} \]

Partition \( p = p_0(x,z) + p'(x,y,z,t) \) into hydrostatic part \( p_0 = p_{00} - p_{00} z \) and perturbation \( p' \).

Then
\[ \left[ \frac{d\rho_0}{dt} \right] = \left[ \frac{\partial p_0}{\partial z} \right] = \frac{U}{H} \] (using horizontal mom eqn \( \frac{d\rho'}{dt} = -\frac{1}{\rho} \frac{dp'}{dt} \) to scale \( p' \))