Fluid eqns:

Continuity: \( \frac{1}{\rho} \frac{D\rho}{Dt} + \nabla \cdot \mathbf{u} = 0 \)  \( \text{(1)} \)

Hor. mom. \( \rho \frac{D\mathbf{u}}{Dt} = -\nabla p' \)  \( \text{(2)} \)

Vert. mom. \( \rho \frac{Dw}{Dt} = -\frac{\partial p'}{\partial z} - \rho'g \)  \( \text{(3)} \)

The scaling argument is simplest if we assume the Boussinesq approximation, replacing (1) by the incompressibility condition

\( \nabla \cdot \mathbf{u} = 0 \)  \( \text{(1')} \)

This automatically requires slow flow and a shallow fluid layer, with small relative density variations. However, the arguments can be patched up for a deep fluid layer too with more effort, and can be extended to rotating fluids.

**Scaling:**

\( \text{(1')} \Rightarrow \left[ \frac{\partial w}{\partial z} \right] = \left[ -\nabla_h \cdot \mathbf{u}_h \right] \leq \frac{U}{L} \Rightarrow W \ll U H \)

\( \text{(2')} \Rightarrow \left[ \rho \frac{D\mathbf{u}_h}{Dt} \right] = \left[ \rho_0 \left( \frac{2}{\partial t} + \mathbf{u} \cdot \nabla + w \frac{\partial}{\partial z} \right) \mathbf{u}_h \right] \)

\[ = \rho_0 \max \left( \frac{1}{T}, \frac{U}{L}, \frac{W}{H} \right) \cdot U \]

\[ = \rho_0 \frac{U}{T} \Rightarrow \nabla p' \approx \frac{U}{L} \]

\[ \Rightarrow \left[ p' \right] = \rho_0 \frac{U}{T} \frac{L}{T} \]

\( \text{(3):} \left[ \rho \frac{Dw}{Dt} \right] = \rho_0 \frac{W}{T} \)

\[ \left[ -\partial p'/\partial z \right] = \frac{\left[ p' \right]}{H} = \rho_0 \frac{U L}{T H} \]

\[ \Rightarrow \left[ \rho \frac{Dw}{Dt} \right] = \rho_0 \frac{W}{T} \]

\[ \left[ -\partial p'/\partial z \right] = \frac{\rho_0 \frac{W}{T}}{\rho_0 \frac{U L}{T H}} = \frac{W}{U} \cdot \frac{H}{L} \leq \left( \frac{H}{L} \right)^2 \ll 1 \]

thus the dominant balance in vertical momentum equation is

\[ \frac{\partial p'}{\partial z} \approx -\rho'g \quad \text{(hydrostatic approximation)} \Rightarrow \left[ p' \right] = \left[ p' \right]/gH \]
If \( H/L < 1 \), the hydrostatic approximation makes the use of pressure \( p \) as a vertical coord. attractive; its main virtue is to simplify the mass continuity eqn. in the atmosphere, where the Boussinesq approximation is not valid for deep flows.

**Key features**

1. **On a const. pressure surface** \( \frac{dp}{dt} = 0 = \frac{2p}{3x} \frac{dx}{dz} + \frac{dp}{dz} \). Thus

   \[
   \frac{\delta z}{\delta x} = -\frac{\delta p/\delta x}{\delta p/\delta z} = \left[ \frac{p/\rho}{p/\rho} \right] \frac{\delta z}{\delta x} \approx \frac{H}{L} \ll 1 \text{ for small aspect ratios.}
   \]
   
   So \( \delta z/\delta x \) quasi-horizontal, thus is assumed in deriving (3)+(4) below.

2. \( \hat{u}_h = \frac{\delta x}{\delta t} \) is unaltered. The new vertical velocity is \( \omega = \frac{dp}{dz} \).

3. The mass \( \delta m \) between the \( p \) and \( p+\delta p \) surfaces in a cylinder of cross-section \( \delta A \) is

   \[
   \delta m = p \delta z \delta A = \frac{\delta p}{\delta A} \delta A \quad \text{(constant of mass)}
   \]

   \[
   \frac{D}{Dt} \delta m = 0 = \frac{1}{\delta m} \frac{D}{Dt} \delta m
   \]

   \[
   = \frac{1}{\delta m} \frac{D}{Dt} \delta p + \frac{1}{\delta A} \frac{D}{Dt} \delta A
   \]

   \[
   = \frac{1}{\delta m} \left[ \frac{D\hat{u}_h}{Dt} - \frac{D\delta p}{Dt} \right] + \nabla_h \cdot \nabla_h \dot{v}_h
   \]

   \[
   = \nabla_h \cdot \hat{v}_h + \frac{\delta \omega}{\delta p} \frac{\delta p}{\delta \omega}
   \]

   (Looks "incompressible" but really isn't)

4. The hor. PGF is \( \frac{-1}{\delta x} \frac{\partial^2 \delta z}{\partial x^2} \). But \( \partial^2 \delta z/\partial x^2 = -\left( \frac{\delta^2 z}{\delta x^2} \right) \), \( \frac{2\delta z}{\delta x} = \frac{\partial^2 \delta z}{\partial x^2} \).

5. **Hydrostatic balance is** \( \frac{\delta z}{\delta p} = -\frac{1}{\rho g} \Rightarrow \frac{\partial \phi}{\partial p} = -\frac{1}{\rho} \)

Combine to get

- **Mom:** \( \frac{D\hat{u}_h}{Dt} = -\nabla_h \phi \)

- **Cont.:** \( \nabla_h \cdot \hat{v}_h + \frac{\partial \omega}{\partial p} \frac{\partial p}{\partial \omega} = 0 \)

**Fluid Eqns in pressure coords**

\( H \ll L \)

**Eqn of state, thermo as before.**

**Disadvantages:** Not applicable to flows with \( H \approx L \) (convection, turbulence)

BCs are subtle. At a flat surface \( w = 0 \) but \( \phi \) may vary, and we instead impose \( \phi(x,y,p,t) = gz \) to determine \( p_s \) from the evolution of the density field, and \( \omega_s = \frac{dp_s}{dt} \), since surface parcels stay on surface.

\( \sigma = \frac{p}{p_s} \) (sigma-coords) are commonly used in numerical applications in atm. sci.