Ekman Layers (CR Ch. 5), Gill Ch. 9)

- Frictional boundary layer in a low Ro rotating fluid (at solid surface or air-water interface). By

\[
\frac{Du}{Dt} + fK \times \hat{u}_h = - \frac{1}{\rho} \frac{\partial p}{\partial y} + \frac{\omega^2}{\rho} \hat{u}_h
\]

For laminar flow in the interior of the fluid, with \(Ro = 1\) and \(H \ll L\),

\[
\frac{[\rho \nabla^2 u_h]}{[LHS]} = \frac{[\rho \nabla^2 \hat{u}_h]}{[fK \times \hat{u}_h]} = \frac{\omega U}{\nu H} = \frac{z}{H^2} = \text{Ek}
\]

For air, \(z = 14 \times 10^5 \text{ m}^2 / \text{s}\). If \(H = 10 \text{ km}\), \(Ek \approx 10^{-9}\) Thus molecular viscosity in unimportant in interior of fluid except where there is turbulence.

However, friction must become important where fluid adjusts to a boundary. For a rigid horizontal boundary, \(u_b = 0\). There is a thin “boundary layer” in which this adjustment takes place in which \(\frac{u}{H} \ll 1\)

\[
l \approx \frac{[\rho \nabla^2 u_h]}{[fK \times \hat{u}_h]} \approx \frac{\rho U}{f} \frac{g}{H} \Rightarrow \frac{d}{H} \approx \left(\frac{R_e}{H}\right)^2, \text{Ek} = \left(\frac{R_e}{H}\right)^2
\]

For air, laminar Ekman layer depth is 0.5 m with \(f = 10^{-9}\). For water, is 0.15 m and 0.5 min with \(f = 6 \times 10^{-9}\) (SL: 33 rpm).

Observed wind-driven BL’s are 0 (200-1000 m) deep in atm, 0 (5-30 m) deep in ocean.

Bottom Ekman layer in steady homogeneous flow (Gill 9.6 has a terse discussion)

Idealizations: steady (\(\frac{\partial u}{\partial t} = 0\)), horizontal homogeneous (\(\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0\)), f-plane, constant density \(\rho\), spatial uniform horizontal pressure gradient force \(\nabla p\).

Look for solution \((u(z), v(z))\) with:

\(u(0) = v(0) = 0\) (No slip at bottom boundary \(z = 0\))

\[-fu = -\frac{1}{\rho} \frac{\partial p}{\partial x} \equiv f u_b \text{ and } fu = -\frac{1}{\rho} \frac{\partial p}{\partial y} \equiv f u_b \text{ as } z \to \infty \]  (No frictional effects far from boundary)

Thus \(\frac{Du}{Dt} = \frac{\partial u}{\partial t} + \frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial y} + \frac{\omega^2}{\rho} u = 0\)

and \(\nabla^2 u = \frac{d^2 u}{dz^2}\) similarly for \(v\). Thus, hor. mom. eqns become:

\[
\frac{Du}{Dt} - fu = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{\omega^2}{\rho} u \Rightarrow -f(u - u_b) = \omega \frac{d^2 u}{dz^2}
\]

\[
\frac{Dv}{Dt} + fu = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \frac{\omega^2}{\rho} v \Rightarrow -f(u - u_b) = \omega \frac{d^2 v}{dz^2}
\]
Setting \( s = u \text{in} \) and \( s_y = v \) \( v \text{in} \), we find (1) + i(2) \[ \Rightarrow \]
\[ \frac{d^2 s}{dz^2} = -f_v (V - V_y) + i (u - u_y) = i s_s \]

The homogeneous eqn has characteristic polynomial \( z^2 - i f = 0 \Rightarrow \lambda = \pm \frac{i}{2} \]

The positive root \( \lambda \) has positive real part so \( e^{\lambda z} \to \infty \) as \( z \to \infty \), inconsistent with BC. In addition, \( s = s_y \) is an inhomogeneous soln by inspection, so

\[ s(x) = s_y \pm \frac{d}{i} e^{i x} \]

Noting that \( r \) is \((1+i) \frac{x}{2} = \frac{x}{2} \), we see \( s(x) = -s_y \) and

\[ s(x) = s_y \left( 1 - e^{-(1+i) x} \right) \]

Setting \( s_y = u \) (positive real) corresponding to eastward geostrophic wind, we see

\[ u + i v = u \left( 1 - e^{-(i) x} \right) \Rightarrow u(x) = u \left( 1 - e^{-x} \cos x \right), v(x) = e^{-x} \sin x \]

(1) Near \( x = 0 \), \( u(x) \) and \( v(x) \) both \( \approx x \) so surface wind is \( 45^\circ \) to left of geostrophic,
(2) and surface wind stress of air on surface (equal and opposite to drag at surface on air):

\[ \frac{\tau}{\rho} = \frac{d}{d t} \frac{d u}{d z}(0) = \rho \frac{d^2}{d z^2} (u + v) \]

Net frictional transport:

\[ \frac{\partial}{\partial t} \int_0^Z \rho(x, y, z) d z = \rho \frac{d}{d t} \left[ \int_0^Z \left( e^{-x} \cos x + e^{-z} \sin z \right) d z \right] \]

Note \( \int_0^Z e^{-x} dx = \frac{e^{-x}}{-1} \)

For a general flow direction \( u \) \( v \):

\[ \frac{\partial}{\partial t} \int_0^Z \rho(x, y, z) d z = \rho \frac{d}{d t} \left[ \int_0^Z \left( u(x, y) \frac{\partial}{\partial x} + v(x, y) \frac{\partial}{\partial y} \right) d z \right] \]

Non-uniform currents + Ekman pumping

Now suppose \( u_y \approx u_y(x, y) \) and \( v \approx v(x, y) \). For a truly homogeneous fluid this is the only way horizontal inhomogeneity that can be supported. Suppose \( u_y, v_y \) vary on a lengthscale \( L \gg \delta \). Then locally frictional BC eqns look as before with \( u_y, v_y \) local values. Note that the frictional flow is divergent even though \( (u_y, v_y) \) is not:

\[ D = \frac{\partial}{\partial x} (u_x + \frac{\partial}{\partial y} (u_y + \frac{\partial}{\partial x} u_y) + \frac{\partial}{\partial y} (u_y - v_y) \]

This must result in a upwelling velocity

\[ \int_0^Z \rho \frac{d}{d z} (u_x + \frac{\partial}{\partial y} (u_y + \frac{\partial}{\partial x} u_y) + \frac{\partial}{\partial y} (u_y - v_y) \]

Effect on interior

\[ \frac{d}{d t} \frac{1}{\rho d x} \int_0^H \frac{d}{d z} \]

Ekman pumping
The Ekman Layer (figures from Cushman-Roisin)

Figure 5-2 The velocity spiral in the bottom Ekman layer. The figure is drawn for the Northern Hemisphere ($f > 0$), and the deflection is to the left of the current above the layer. The reverse holds for the Southern Hemisphere.

Figure 5-3 Divergence in the bottom Ekman layer and compensating downwelling in the interior. Such situations arise in the presence of an anticyclonic gyre in the interior. Similarly, interior cyclonic motion causes convergence in the bottom Ekman layer and upwelling in the interior.

Figure 5-4 Structure of the surface Ekman layer. The figure is drawn for the Northern Hemisphere ($f > 0$), and the deflection is to the right of the surface stress. The reverse holds for the Southern Hemisphere.

Figure 5-6 Ekman pumping in an ocean subject to sheared winds (Northern Hemisphere).
Fig. 9.2. Wind hodograph for the lowest kilometre as measured by Dobson (1914) (dashed line) compared with Ekman spiral (full line). Figures on curves are heights in metres.

Figure 5.7 Comparison between observed currents below a drifting ice floe at 84.3°N and theoretical predictions based on an eddy viscosity $v = 2.4 \times 10^{-1} \text{ m}^2/\text{s}$. (Reprinted from *Deep-Sea Research*, 13, Kenneth Hunkins, Ekman drift currents in the Arctic Ocean, p. 614, copyright 1966, with kind permission from Pergamon Press Ltd, Headington Hill Hall, Oxford OX3 0BW, UK.)