Shallow Water Equations (CR4.2-3)

Bores: Hydrostatic approx: \( p(x,y,t) = p_0g(h(x,y,t) - z) \)

\[
\frac{\partial p}{\partial x} = p_0g \frac{\partial h}{\partial x}, \quad \frac{\partial p}{\partial y} = p_0g \frac{\partial h}{\partial y}
\]

\( \frac{\partial h}{\partial x} + f \frac{\partial v}{\partial y} = 0 \)

Consistent to assume \( u, v \) ind of \( x \)

No friction:

\[
\frac{Du}{Dt} - f v = -g \frac{\partial h}{\partial x}
\]

\[
\frac{Du}{Dt} + f u = -g \frac{\partial h}{\partial y}
\]

\[
\frac{h - z_b}{\partial t} + w(h) - w(z_b) = (h - z_b) \frac{2u}{\partial x} = -(h - z_b) \left( \frac{3u}{\partial x} + \frac{2v}{\partial y} \right)
\]

Flat bottom:

\[
\frac{Dh}{Dt} + h \left( \frac{2u}{\partial x} + \frac{2v}{\partial y} \right) = 0
\]

Low \( R_o \) flow (CR4.1)

Let \( \begin{bmatrix} u & v \end{bmatrix} = \begin{bmatrix} U & \frac{U^2}{L} \end{bmatrix} \)

Then \( \begin{bmatrix} [Du/\partial t] \end{bmatrix} = \begin{bmatrix} \frac{U^2}{L} \end{bmatrix} = \frac{U}{T} = L/U \)

If \( R_o \ll 1 \),

\[ f \frac{2u}{\partial x} - f v = -g \frac{\partial h}{\partial x} \]

(geostrophic balance)

\[ f u \approx -g \frac{\partial h}{\partial x} \]

In fact \( \frac{Du}{\partial t} = \frac{U}{T} \)

Coriolis and pressure gradient accelerations balance.

Note this is diagnostic, this leading balance cannot predict changes in flow. Also note that if \( \Gamma \) varies,

\[
\frac{\partial \Gamma}{\partial t} = O(R_o) \approx 0
\]

Since for nearly inviscid version.

\( \begin{bmatrix} 2 \frac{\partial \Gamma}{\partial x} \end{bmatrix} = 0 \)

Vorticity + PV (CR4.4), Gill 7.10

Let \( \Gamma = \frac{2u}{\partial x} - \frac{2v}{\partial y} \)

Then \( (1-2) \rightarrow \)

\[
\frac{2}{\partial x} \left( \frac{2u}{\partial x} - \frac{2v}{\partial y} \right) + f \left( \frac{2u}{\partial x} + \frac{2v}{\partial y} \right) = 0
\]

\[
I = \frac{2u}{\partial x} + \frac{2v}{\partial y} \left( \frac{2u}{\partial x} - \frac{2v}{\partial y} \right) + \frac{u}{\partial x} \left( \frac{2u}{\partial x} + \frac{2v}{\partial y} \right) + \frac{v}{\partial y} \left( \frac{2u}{\partial x} - \frac{2v}{\partial y} \right)
\]

\[
= \frac{2u}{\partial x} + \frac{2v}{\partial y} \left( \frac{2u}{\partial x} - \frac{2v}{\partial y} \right) + \frac{u}{\partial x} \left( \frac{2u}{\partial x} + \frac{2v}{\partial y} \right) + \frac{v}{\partial y} \left( \frac{2u}{\partial x} - \frac{2v}{\partial y} \right)
\]

\[
= \frac{\partial}{\partial t} \left( u + v \right) + v \frac{\partial}{\partial y} \left( \frac{\partial}{\partial x} \left( u + v \right) \right) + u \frac{\partial}{\partial x} \left( \frac{\partial}{\partial y} \left( u + v \right) \right)
\]