1.a. Given:

\[
\frac{H}{L} << 1 \\
R_o = \frac{U}{H} << 1
\]

low Mach number
characteristic density \( \rho_{oo} = [\rho] \)

Define \( \rho = \rho_o(z) + \rho'(x,t) \), \( p = p_o(z) + p'(x,t) \) such that \( \rho_o \) and \( p_o \) are in hydrostatic balance.

Let \( \frac{\partial \rho}{\partial t} = \frac{1}{T} = \frac{U}{L} \)

Assume \( \rho' << \rho_{oo} \)

\( W = U \frac{H}{L} \)

Scaling the material derivative of velocity:

\[
\begin{align*}
\frac{Du}{Dt} &= \frac{\partial u_H}{\partial t} + \frac{\partial w}{\partial t} + u_H \cdot \nabla u_H + w \frac{\partial w}{\partial z} \\
\begin{bmatrix}
\frac{Du}{Dt}
\end{bmatrix} &= \frac{U^2}{L} + \frac{U^2}{L} \frac{H}{L} + \frac{U^2}{L} \frac{H}{L} + \frac{U^2}{L} \frac{H}{L}
\end{align*}
\]

Since \( \frac{H}{L} << 1 \), the second and fourth terms are negligible, leaving

\[
\begin{bmatrix}
\frac{Du}{Dt}
\end{bmatrix} = \frac{U^2}{L}
\]

Scaling the horizontal momentum equation:

\[
\rho \frac{Du_H}{Dt} - \rho f(v_i - u_j) = -\nabla_H p'
\]
\[
\frac{\rho_oo}{L} \frac{U^2}{L} + \Omega U = \frac{[p']}{L} \\
[p'] = \rho_oo U^2 + \rho_oo L\Omega U
\]

Now scaling the **vertical momentum** equation:

\[
\begin{align*}
\rho Dw &= -\frac{\partial p'}{\partial z} - \rho' g \\
\rho_oo \frac{WU}{L} &= \frac{[p']}{H} + [\rho']g \\
\rho_oo U^2 \frac{H^2}{L^2} &= \rho_oo U^2 + \rho_oo L\Omega U + [p']gH
\end{align*}
\]

Dividing the LHS from the first term on the RHS:

\[
\frac{\rho_oo U^2 H^2}{\rho_oo U^2} = \frac{H^2}{L^2} << 1
\]

So the term on the LHS is negligible. Dividing the first term on the RHS from the second (both are parts of \([p']\)):

\[
\frac{\rho_oo U^2}{\rho_oo L\Omega U} = \frac{U}{fL} = R_o << 1
\]

Now all we have left in the vertical momentum balance is:

\[
[p'] = \rho_oo L\Omega U = [\rho']gH
\]

1.b. Apply the scalings

i) Midlatitude storms

\[
\begin{align*}
[p'] &= [\rho']gH \\
[p'] &= (0.05)(10)(10^4) = 5000 \text{ Pa} = 50 \text{ mbar} \\
U &= \frac{[\rho']gH}{\rho_oo L\Omega} \\
U &= \frac{5000}{(1)(2 \times 10^9)(10^{-4})} = 25 \text{ m s}^{-1}
\end{align*}
\]

ii) The Gulf Stream
\[ [p'] = [\rho'] gH, \quad [\rho'] = \rho_{oo}\alpha \Delta T = (1000)(1.7 \times 10^{-4})(10) = 1.7 \text{ kg m}^{-3} \]

\[ [p'] = (1.7)(10)(10^3) = 1.7 \times 10^4 \text{ Pa} \]

\[ U = \frac{[\rho'] gH}{\rho_{oo} L \Omega} \]

\[ U = \frac{1.7 \times 10^4}{(1000)(10^5)(10^{-4})} = 1.7 \text{ m s}^{-1} \]

1.c. First find \([u_g]\) from the definition of geostrophic velocity:

\[
\begin{align*}
\rho_{oo} f k \times u_g &= -\nabla_H p' \\
\rho_{oo} \Omega [u_g] &= \frac{[p']}{L} \\
[u_g] &= \frac{\rho_{oo} L \Omega U}{\rho_{oo} L} \\
[u_g] &= U
\end{align*}
\]

Now separate the horizontal momentum equation into geostrophic and ageostrophic parts:

\[
\begin{align*}
\rho \frac{Du_H}{Dt} + \rho f k \times u_g + \rho f k \times u_{ag} &= -\nabla_H p' \\
\rho \frac{Du_H}{Dt} &= -\rho f k \times u_{ag} \\
\frac{U^2}{L} &= \Omega [u_{ag}] \\
[u_{ag}] &= \frac{U^2}{\Omega L} = U \frac{R_o}{L}
\end{align*}
\]

1.d. Starting with the Boussinesq continuity equation and separating the horizontal velocity into geostrophic and ageostrophic components:

\[
\begin{align*}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0 \\
\frac{\partial u_g}{\partial x} + \frac{\partial u_{ag}}{\partial x} + \frac{\partial v_g}{\partial y} + \frac{\partial v_{ag}}{\partial y} + \frac{\partial w}{\partial z} &= 0
\end{align*}
\]

Since purely geostrophic velocities have no horizontal divergence:
\[
\frac{\partial u_{ag}}{\partial x} + \frac{\partial v_{ag}}{\partial y} + \frac{\partial w}{\partial z} = 0
\]

\[
\frac{[u_{ag}]}{L} = \frac{W}{H}
\]

\[
W = R_o \frac{H}{L}
\]

Applying the scalings,

i) Midlatitude storms

\[
W = R_o \frac{H}{L} = \frac{U^2 H}{L^2 \Omega}
\]

\[
W = \frac{(25)^2(10)(1000)}{(2 \times 10^6)^2(10^{-4})} = 0.016 \text{ m s}^{-1}
\]

ii) Gulf Stream

\[
W = R_o \frac{H}{L} = \frac{U^2 H}{L^2 \Omega}
\]

\[
W = \frac{(1.7)^2(1000)}{(10^5)^2(10^{-4})} = 0.0029 \text{ m s}^{-1}
\]

2.a. This problem is similar to the example on page 13a of the lecture notes. Instead of a constant forcing, the forcing \( F_x = \frac{\tau}{\rho H} \) increases linearly, then becomes constant.

\[
\tau(t) \begin{cases} 
\frac{\tau_0}{T}; & 0 \leq t \leq T \\
\frac{\tau_0}{T}; & t > T 
\end{cases}
\]

The horizontal momentum equations are given by:

\[
\frac{\partial u}{\partial t} - fv = \frac{\tau}{\rho H}
\]

\[
\frac{\partial v}{\partial t} + fu = 0
\]

Setting \( s = u + iv \):
\[
\frac{ds}{dt} = \frac{du}{dt} + i \frac{dv}{dt} = (fv + \frac{\tau}{\rho H} + i(-fu))
= \frac{\tau}{\rho H} - if(u + iv)
\rightarrow \frac{ds}{dt} + ifs = \frac{\tau}{\rho H}
\]

For time \(0 \leq t \leq T\), the forcing term is a linear function of time, resulting in a nonhomogenous ODE. The solution can be obtained by splitting it into complementary and particular parts such that \(s = s_c + s_p\). The complementary part of the solution is the solution to the ODE with no forcing:

\[
\frac{ds_c}{dt} + ifs_c = 0
\]
\[
s_c = ce^{-ift}
\]

The particular part of the solution can be found by the method of undetermined coefficients. Since the forcing is linear, we guess a linear solution \(s_p = at + b\):

\[
\frac{ds_p}{dt} + ifs_p = \left(\frac{\tau_o}{\rho H}\right) \frac{t}{T}
\]
\[
a + ifs(at + b) = \left(\frac{\tau_o}{\rho H}\right) \frac{t}{T}
\]

\[
\rightarrow a = \left(\frac{\tau}{\rho H}\right) \frac{1}{ift} \quad \frac{\tau_o}{\rho H}
\]
\[
b = \left(\frac{\tau_o}{\rho H}\right) \frac{1}{f^2T}
\]

The solution therefore has the form:

\[
s = ce^{-ift} + \left(\frac{\tau_o}{\rho H}\right) \frac{t}{ift} + \left(\frac{\tau_o}{\rho H}\right) \frac{1}{f^2T}
\]

The constant \(c\) can be found by applying the initial condition \(s(0) = 0\), leaving an exact solution

\[
s = -\left(\frac{\tau_o}{\rho H}\right) \frac{1}{Tf^2} \left[\cos (ft) - i \sin (ft) - i \left(\frac{\tau_o}{\rho H}\right) \frac{t}{fT} + \left(\frac{\tau_o}{\rho H}\right) \frac{1}{f^2T}\right]
\]
Therefore $u$ and $v$ are

\[
\begin{align*}
    u &= \left( \frac{\tau_0}{\rho H} \right) \frac{1}{f^2T} \left[ \cos (ft) \right] \\
    v &= \left( \frac{\tau_0}{\rho H} \right) \frac{1}{fT} \left[ \frac{1}{f} - t \right]
\end{align*}
\]

for time $0 \leq t \leq T$.

By integrating either $s$ or $u$ and $v$, and using the initial condition that $x, y = 0$ at $t = 0$, the positions $x$ and $y$ can be found:

\[
\begin{align*}
    x &= \left( \frac{\tau_0}{\rho H} \right) \frac{1}{f^2T} [t - \sin (ft)] \\
    y &= \left( \frac{\tau_0}{\rho H} \right) \frac{1}{f^2T} \left[ \frac{1}{f} - t - \frac{1}{2} f^2 + \cos (ft) \right]
\end{align*}
\]

for time $0 \leq t \leq T$. At time $t > T$, the forcing is constant:

\[
\frac{ds}{dt} + ifs = \frac{\tau_0}{\rho H}
\]

\[
\rightarrow s = \left( \frac{\tau_0}{\rho H} \right) \frac{1}{if} \left[ 1 - e^{-if(t-T)} \right]
\]

At time $T$, the velocity and position are known from the solution for time $0 \leq t \leq T$. Let us define $u_T, v_T, x_T, y_T$ as the velocity and position at time $T$. Applying the initial condition $s(T) = s_T = u_T + iv_T$ gives the exact solution for $s$:

\[
\begin{align*}
s &= \left( \frac{\tau_0}{\rho H} \right) \frac{1}{if} \left[ 1 - (1 - \left( \frac{\rho H}{\tau_0} \right) ifs_T) e^{-if(t-T)} \right] \\
    &= -\frac{i}{f} \left( \frac{\tau_0}{\rho H} \right) + i \left( \frac{\tau_0}{\rho H} \right) \cos [f(t-T)] + \frac{1}{f} \left( \frac{\tau_0}{\rho H} \right) \sin [f(t-T)] + u_T \cos [f(t-T)] \\
    &\quad - iu_T \sin [f(t-T)] + iv_T \cos [f(t-T)] + v_T \sin [f(t-T)]
\end{align*}
\]

\[
\begin{align*}
    \rightarrow u &= \frac{1}{f} \left( \frac{\tau_0}{\rho H} \right) \sin [f(t-T)] + u_T \cos [f(t-T)] + v_T \sin [f(t-T)] \\
    v &= -\frac{1}{f} \left( \frac{\tau_0}{\rho H} \right) + \frac{1}{f} \left( \frac{\tau_0}{\rho H} \right) \cos [f(t-T)] - u_T \sin [f(t-T)] + v_T \cos [f(t-T)]
\end{align*}
\]
for time $t > T$. Integrating gives the position in $x$ and $y$:

\[
x = x_T + \frac{v_T}{f} + \frac{1}{f^2} \left( \frac{\tau_o}{\rho H} \right) - \frac{1}{f^2} \left( \frac{\tau_o}{\rho H} \right) \cos [f(t - T)] + \frac{u_T}{f} \sin [f(t - T)] - \frac{v_T}{f} \cos [f(t - T)] \]
\[
y = y_T - \frac{u_T}{f} + \frac{1}{f} \left( \frac{\tau_o}{\rho H} \right) (t - T) + \frac{1}{f^2} \left( \frac{\tau_o}{\rho H} \right) \sin [f(t - T)] + \frac{u_T}{f} \cos [f(t - T)] + v_T f \sin [f(t - T)]
\]

for time $t > T$.

2.b. The complex velocity for time $t > T$ can be written in the form

\[
s = -i \left( \frac{\tau_o}{\rho H} \right) \frac{1}{f} + \left[ s_T + i \left( \frac{\tau_o}{\rho H} \right) \frac{1}{f} \right] e^{-i f(t - T)}
\]
\[
= -i \left( \frac{\tau_o}{\rho H} \right) \frac{1}{f} + \left( \frac{\tau_o}{\rho H} \right) \frac{1}{f f^2} \left[ 1 - \cos (fT) + i \sin (fT) \right] e^{-i f(t - T)}
\]

The first term on the right hand side shows that there is a constant component of the velocity in the $-iy$ direction, or 90 degrees to the right of the wind. The Ekman drift is $v_E = - \left( \frac{\tau_o}{\rho H} \right) \frac{1}{f}$. This drift is maintained by a balance between the wind forcing and the time averaged Coriolis force.

The oscillatory component of the velocity has a magnitude which depends on $T$, as seen in the first figure. The more slowly the wind forcing is applied, the more closely the Coriolis force balances the wind forcing at any time $0 \leq t \leq T$. If, when the wind forcing becomes constant, it is almost completely balanced by the Coriolis force, then the particle can only travel at a velocity close to $v_E$ in a nearly straight line perpendicular to the two forces.

The trajectory of a particle starting at the origin is shown in the second figure for three different cases. The parameters are $f = 10^{-4} \text{ rad/s}$, $\rho = 1000 \text{ kg/m}^3$, $H = 100 \text{ m}$ and $\tau = 0.1 \text{ Pa}$. 

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