ATMS 581 / AMATH 586 — Spring 06

Homework 3 – Take-Home Midterm

Work Independently!

Note: when reporting the truncation error in these exercises, give the full expressions for the lowest-order error terms, not simply the order of the truncation error.

1. Consider the shallow-water equations, linearized about a state at rest,
\[
\frac{\partial u}{\partial t} + g \frac{\partial \eta}{\partial x} = 0, \quad \frac{\partial \eta}{\partial t} + H \frac{\partial u}{\partial x} = 0.
\]
Prove, without doing a Von-Neumann stability analysis, that the following finite-difference approximation to the preceding system must be unstable
\[
\frac{u_{j}^{n+1} - u_{j}^{n}}{\Delta t} + g \left( \frac{\eta_{j}^{n} - \eta_{j-1}^{n}}{\Delta x} \right) = 0,
\]
\[
\frac{\eta_{j}^{n+1} - \eta_{j}^{n}}{\Delta t} + H \left( \frac{u_{j}^{n} - u_{j-1}^{n}}{\Delta x} \right) = 0.
\]

2. Approximate solutions to the constant-wind-speed advection equation
\[
\frac{\partial \psi}{\partial t} + c \frac{\partial \psi}{\partial x} = 0
\]
are sought using the finite-difference scheme
\[
\frac{\phi_{j}^{n+1} - \phi_{j}^{n}}{\Delta t} + \frac{c}{2} \left( \frac{\phi_{j+1}^{n+1} - \phi_{j-1}^{n+1}}{\Delta x} + \frac{\phi_{j+1}^{n} - \phi_{j}^{n}}{\Delta x} \right) = 0.
\]
Determine the truncation error and the stability properties of this method.

3. Derive the modified equation that is approximated through third-order by the leapfrog-time centered-space approximation to the constant wind-speed advection equation (1)
\[
\frac{\phi_{j}^{n+1} - \phi_{j}^{n-1}}{2\Delta t} + \frac{c}{2} \frac{\phi_{j+1}^{n} - \phi_{j-1}^{n}}{2\Delta x} = 0.
\]
Compare this with the modified equation for the Lax-Wendroff approximation to the same continuous problem, which is
\[
\frac{\partial \psi}{\partial t} + c \frac{\partial \psi}{\partial x} = -(1 - \mu^2) \frac{c(\Delta x)^2}{6} \frac{\partial^3 \psi}{\partial x^3} - \mu(1 - \mu^2) \frac{c(\Delta x)^3}{8} \frac{\partial^4 \psi}{\partial x^4},
\]
where \( \mu = c\Delta t/\Delta x \). Discuss whether the behavior of these two schemes, as illustrated in Fig. 2.19 on page 97 of Durran, is consistent with the leading-order error terms in each scheme’s modified equation.

Due Wednesday May 10th