5.1 Turbulence

5.1.1 Boussinesq Approximation

To predict flow within the boundary layer we must solve a complete set of conservation equations for the five principal variables $u, v, w, \theta,$ and $p$. Because the boundary layer is relatively thin, the equation are often simplified using the Boussinesq approximation, which treats the density as constant except in the buoyancy force term. An example is the vertical momentum equation:

$$\frac{Dw}{Dt} = -\frac{1}{\rho_o} \frac{\partial p}{\partial z} + g \frac{\theta}{\theta_o} + F_{rz}. \tag{1}$$

where $p$ and $\theta$ have been expanded as, for example, $\theta_{tot} = \theta_o(z) + \theta(x, y, z, t)$ with $\theta_o$ equal to the basic state density and $\theta$ equal to the departure from the basic state and $\rho_o = \text{const}.$

The buoyancy force resembles the one in Eq 2.51, but the variables here are different, which is why Eq 1 (also Holton Eq 5.4) still has a pressure gradient force appearing explicitly while Eq 2.51 does not. Recall that in Eq 2.51 we dealt with buoyancy of a parcel displaced vertically and adiabatically from a point where it was at equilibrium with its environment.

In Eq (1), the buoyancy force deals with the buoyancy of the local environment compared to the basic state. The buoyancy force in Eq 1 is derived on page 197, which is the next time you will see the Boussinesq approximation. Don’t sweat it now.

Note that the friction force is not neglected in the boundary layer equations. $F_r$ is for molecular friction, not turbulence. Where is the turbulence? It is lurking in the Lagrangian derivatives as advection, which is nonlinear. For example,

$$\frac{Dw}{Dt} = \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z}. $$

Products of the principal variables in the last three terms are strongly influenced by turbulence. Although there are five equations and five unknowns, the nonlinear advection terms in this form make the five Boussinesq equations hard to solve.
5.1.2 Reynold’s averaging

The situation is helped by using Reynold’s averaging. Let

\[ w = \overline{w} + w' \]

where \( \overline{w} \) is the slowly varying part of \( w \) (ie slow compared to the timescale of eddy motions, so maybe daily varying) and \( w' \) is the departure from \( \overline{w} \) (ie the turbulent part). The overline could represent a running average. See the matlab script reynolds.m downloadable from the class schedule for an example.

Note that

\[ \overline{w'} = 0 \quad \text{and} \quad \overline{w' \theta'} = \overline{w'} \overline{\theta'} = 0 \]

so

\[ \overline{w \theta} = \overline{w} + \overline{w'} \overline{\theta} + \overline{\theta'} = \overline{w \theta} + \overline{w' \theta'} \]

\( \overline{w' \theta'} \) is the covariance. If on average turbulent eddies have \( w' \) and \( \theta' \) both either both positive or negative, then the covariance is positive, see Fig 5.1

\( \overline{w' \theta'} \) is a vertical flux of heat from turbulent eddies (when multiplied by \( c_p \))

\( \overline{w' u'} \) is a vertical flux of u-momentum (or a \( \hat{i} \)-direction flux of vertical momentum)

Reynolds averaging for the vertical momentum equations gives us

\[
\frac{D\overline{w}}{Dt} = \left( \frac{\partial}{\partial t} + \overline{u} \frac{\partial}{\partial x} + \overline{v} \frac{\partial}{\partial y} + \overline{w} \frac{\partial}{\partial z} \right) \overline{w} = \frac{-1}{\rho_o} \frac{\partial \overline{p}}{\partial z} + \frac{g}{\theta_o} \overline{\theta} - \left[ \frac{\partial \overline{u' w'}}{\partial x} + \frac{\partial \overline{v' w'}}{\partial y} + \frac{\partial \overline{w' w'}}{\partial z} \right] - F_{rz}.
\]

Inbetween the square brackets are the turbulent flux terms \( F_{rz} \), specifically here it is the divergence of the turbulent (or eddy) vertical momentum flux.

Using Reynold’s averaging we can rewrite each of the five Boussinesq equations similar to the vertical momentum equation, giving us five equations with five unknown mean variables \( \overline{u}, \overline{v}, \overline{w}, \overline{\theta}, \) and \( \overline{p} \) plus all the unknown turbulent fluxes. “Closure” assumptions must be made to relate the turbulent fluxes to the mean variables, see section 5.3.
5.2 Turbulent Kinetic Energy

By its nature, turbulence tends to cause its own energy to flow towards small scales where the energy is dissipated by molecular viscous diffusion. Turbulence is created primarily by vertical shear in the horizontal wind field and by surface heating causing an unstable lapse rate. The two weeks prior to Thanksgiving were an excellent example of very little turbulence. Boundary layer pressure gradients were weak due to the pressure ridge early on and eventual closed surface high parked over Washington. The atmosphere was also remarkably stable owing to weak solar insolation and low fog. The fog was a result of the stable temperature profile (and windless conditions), promoting the conditions that formed it.

5.3 Boundary Layer Momentum Equations

Often we neglect the horizontal derivatives of the turbulent fluxes, greatly simplifying the Boussinesq B.L. equations. In this section, we just focus on describing the horizontal circulation. Neglecting molecular viscous friction the equations are

\[ \frac{D\bar{u}}{Dt} = -\frac{1}{\rho_o} \frac{\partial p}{\partial x} + f\bar{v} - \frac{\partial \bar{u}'w'}{\partial z} \]
\[ \frac{D\bar{v}}{Dt} = -\frac{1}{\rho_o} \frac{\partial p}{\partial y} - f\bar{u} - \frac{\partial \bar{v}'w'}{\partial z}. \]

Neglecting horizontal acceleration and using the definition of the geostrophic wind in z-coordinates:

\[ f(\bar{v} - \bar{v}_g) - \frac{\partial \bar{u}'w'}{\partial z} \approx 0 \]
\[ -f(\bar{u} - \bar{u}_g) - \frac{\partial \bar{v}'w'}{\partial z} \approx 0. \]

5.3.1 Well Mixed Boundary Layer - Slab

The profiles in Fig 5.2 are typical of atmospheric conditions when the surface is warmer than the overlying air, giving an unstable surface layer and neutral conditions aloft.
We approximate $\overline{w-o}$ and $\overline{w}$ as constants in the mixed layer. Thus we can define Bulk Aerodynamic formulas for the turbulent fluxes at the surface anemometer height (2m above the surface):

$$(u'w')_s = -C_d|\nabla|\overline{u}$$

$$(v'w')_s = -C_d|\nabla|\overline{v}$$

where $\overline{u}$ and $\overline{v}$ are for the mixed layer and $C_d$ is the Drag Coefficient.

Integrating the u-momentum equation gives

$$\int_0^h f(\overline{v} - \overline{v_g}) dz = \int_0^h \frac{\partial(u'w')}{\partial z} dz = \int_{(u'w')_s}^0 du'w'$$

$$f(\overline{v} - \overline{v_g}) h = -(u'w')_s = C_d|\nabla|\overline{u}$$

likewise for v-momentum

$$-f(\overline{v} - \overline{v_g}) h = C_d|\nabla|\overline{v}$$

For the sake of comparing magnitudes, rotate the coordinates so $\hat{i}$ points at $\nabla^*_g$ which makes $\overline{v_g} = 0$. Let $K_s = C_d/f h$:

$$\overline{v} = K_s|\nabla|\overline{u} \quad \text{and} \quad \overline{u} = \overline{v_g} - K_s|\nabla|\overline{u}$$

It is easy to show that $|\nabla|$ is always less than $u_g$

$$|\nabla| = \frac{u_g}{1 + K_s|\nabla|} < u_g$$
Unlike the geostrophic wind, which is perpendicular to the pressure gradient, $\nabla$ points slightly towards the low pressure. $\nabla$ is to the left of the geostrophic wind in the N.H.

5.3.2 The flux-gradient theorem

If the atmosphere is stable or neutral, the slab formulas are not satisfactory because $\pi$ varies with $z$. Instead “K-theory” suggests

$$ (u'u')_s = -K_m \frac{\partial \pi}{\partial z} $$

$$ (v'v')_s = -K_m \frac{\partial \pi}{\partial z} $$

where $K_m$ is the eddy momentum viscosity.

Assuming $K_m > 0$, these equations let the flux of momentum move down the vertical gradient of the horizontal velocity. (Kind of like when heat flows from hot to cold, but here the flux of vertical momentum flows from higher horizontal wind speeds towards lower.)

5.3.4 Ekman Layer

Plug K-theory equations into momentum equations and dropping the overbars (to make the equations neater)

$$ K_m \frac{\partial^2 u}{\partial z^2} + f(v - v_g) = 0 \quad (2) $$

$$ K_m \frac{\partial^2 v}{\partial z^2} - f(u - u_g) = 0 \quad (3) $$

These equations are assumed to apply throughout the B.L. (including the S.L. for now).
Boundary conditions require \( u = v = 0 \) at \( z = 0 \) and \( u \to u_g \) and \( v \to v_g \) at \( z \to \infty \).

Take Eq(1) + \( i \times \) Eq (2), where \( i = \sqrt{-1} \):

\[
K_m \frac{\partial^2 u + iv}{\partial z^2} - if(u + iv) = -if(u_g + iv_g)
\]

Take \( V_g \) independent of \( z \) and rotate axes so \( v_g = 0 \). We find the general solution:

\[
(u + iv) = A \exp \left[ (if/K_m)^{1/2} z \right] + B \exp \left[ -(if/K_m)^{1/2} z \right] + u_g
\]

Apply B.C.’s plus note \( \sqrt{i} = (1 + i)/\sqrt{2} \) and \( f > 0 \) for N.H., we find \( A = 0 \)

\[
(u + iv) = u_g - u_g \exp \left[ -\gamma (1 + i) z \right]
\]

for \( \gamma = \sqrt{f/(2K_m)} \)

Using the identity \( e^{-i\theta} = \cos \theta - i \sin \theta \)

\[
u = u_g \left( 1 - e^{-\gamma z} \cos \gamma z \right) \quad \text{and} \quad v = u_g e^{-\gamma z} \sin \gamma z
\]

At \( \gamma z = \pi \), \( V \) is parallel to \( V_g \) and the \( |V| \approx |V_g| \). Thus we define the Ekman B.L. depth \( D_e = \pi/\gamma \).

As in the slab mixed layer theory above, \( V \) points to the left of \( V_g \) below the depth of the B.L. Check out the matlab script downloadable from the schedule to picture the Ekman spiral in 2 and 3-D.

5.3.5-5.3.6

The main point is that to improve upon Ekman layer theory we need to do a better job in the surface layer in particular. If we take into account of the fact that an eddies size can’t be larger than its distance above the surface, we modify \( K_m \) in the surface layer according to the mixing length hypothesis (see sec 5.3.3)

\[
K_m = (kz)^2 \left| \frac{\partial \pi}{\partial z} \right|
\]
Then we find in the surface layer

$$\overline{u} = \frac{u_*}{k} \ln \frac{z}{z_o}$$

where $k$ = von Karmen’s constant ($\approx 0.4$),
$z_o$ is the roughness length ($\approx 1-4$ cm), and
$u_*^2 = (\overline{u'w'})_s$ is the friction velocity ($\approx 10$ cm/s).

5.4 Secondary Circulations and Spin Down

Drag from turbulent eddies causes cross isobar flow towards low pressure, which results in mass convergence in cyclonic flow and divergence in anticyclonic flow, see Fig 5.6

Recall the derivation on the last exam, convergence of the horizontal flow gives vertical motion.

Again with $v_g = 0$, let

$v$= departure from geostrophic flow $\rho_o v$= mass flux across isobars $M = \text{vertical integral of mass flux in B.L.}$

For a slab M.L. $v$ is independent of $z$ so $M = \rho_o v h$

For the Ekman spiral

$$M = \int_0^{D_e} \rho_o v \, dz = \int_0^{D_e} \rho_o u_g e^{-\gamma / D_e} \sin(\pi z / D_e) \, dz$$