2.1.1 Total differential of a vector in a rotating system

We learn on Holton page 29-30 that the total derivative \((D/Dt)\) of a field variable is the rate of change of the variable following the motion. Here the objective is to derive a relationship between the total derivative of a vector, \(\mathbf{A}\), in inertial and rotating reference frames, to relate \(D_a\mathbf{A}/DT\) to \(D\mathbf{A}/DT\). (The tiny a refers to absolute frame, which is another name for the inertial frame.) Both of these derivatives follow the motion, but as you will see, following the motion of a vector depends on the coordinate system. From the inertial frame, the coordinates in the rotating system are moving, while the inertial coordinates are fixed.

Using Cartesian coordinates in both frames, in an inertial frame the vector may be written:

\[
\mathbf{A} = i'A'_x + j'A'_y + k'A'_z, \tag{1}
\]

and in the rotating:

\[
\mathbf{A} = iA_x + jA_y + kA_z. \tag{2}
\]

Hence

\[
\frac{D_a\mathbf{A}}{Dt} = i\frac{DA'_x}{Dt} + j\frac{DA'_y}{Dt} + k\frac{DA'_z}{Dt} \tag{3}
\]

where we can drop little a on the total derivatives on the right because they are operating on scalars. While if we use rotating coordinates for \(\mathbf{A}\) we use the product rule to get

\[
\frac{D\mathbf{A}}{Dt} = i\frac{DA_x}{Dt} + j\frac{DA_y}{Dt} + k\frac{DA_z}{Dt} + A_x\frac{D_i}{Dt} + A_y\frac{Dj}{Dt} + A_z\frac{Dk}{Dt}. \tag{4}
\]

The last three terms arise because \((i, j, k)\) change direction in space with rotation.

In spherical coordinates, rotation causes longitude \(\lambda\) to change with \(\delta\lambda = \Omega\delta t\) but no change to latitude or height, \(\delta\phi = \delta z = 0\). Hence,

\[
\delta i = \frac{\partial i}{\partial \lambda} \delta \lambda \tag{5}
\]

and in the limit \(\delta t \to 0\),

\[
\frac{D_a i}{Dt} = -\Omega \frac{\partial i}{\partial \lambda}. \tag{6}
\]
\( \mathbf{i} \) is constantly “falling” towards the axis of rotation, which is in the \( \hat{R} \) direction, just like the velocity vector of the ball on the end of a string in Holton Fig 1.5, so

\[
\frac{\partial \mathbf{i}}{\partial \lambda} = \hat{R}.
\]

Note that the magnitude of the rate of change of a unit vector also has unit length. Because

\[
\hat{R} = \mathbf{j} \sin \phi - \mathbf{k} \cos \phi
\]

and

\[
\mathbf{\Omega} = \mathbf{j} \Omega \cos \phi + \mathbf{k} \Omega \sin \phi
\]

after a little algebra we find

\[
\frac{D_a \mathbf{i}}{Dt} = \mathbf{\Omega} \times \mathbf{i}
\]

(7)

\[
\frac{D_a \mathbf{j}}{Dt} = \mathbf{\Omega} \times \mathbf{j} \quad \text{and} \quad \frac{D_a \mathbf{k}}{Dt} = \mathbf{\Omega} \times \mathbf{k}
\]

are harder to show. Fig 2.4 helps visualize them a little. Try to do them on your own.

Finally, putting it all together

\[
\frac{D_a \mathbf{A}}{Dt} = \frac{D \mathbf{A}}{Dt} + \mathbf{\Omega} \times \mathbf{A}.
\]

(8)