Final Assignment Solution

1) Consider a FEM using an arbitrary monotone increasing set of nodes $x_j, j = 0, \ldots, N$, based on piecewise linear chapeau functions $\varphi_j(x)$, each of which is 1 at $x_j$ and 0 at all other nodes:

$$q(x,t) = \sum_{j=0}^{N} q_j(t) \varphi_j(x).$$

The node $x_0 = 0$ is chosen to be the left boundary, so its expansion coefficient $q_0(t)$ is determined by the left BC to be $q_0(t) = \sin(50t)$. The node $x_N$ is chosen to be the right boundary; its expansion coefficient is unknown. Let $\Delta x_{j+1/2} = x_{j+1} - x_j$. As in class, the FEM equations are found by zeroing the projection of the residual onto each basis function that has an unknown expansion coefficient:

$$0 = \langle \varphi_j(x), R(x,t) \rangle = \sum_{n=0}^{N} I_{jn} \frac{d a_n}{d t} + \sum_{n=0}^{N} J_{jn} a_n, j = 1, \ldots, N.$$

The required inner products are easily computed for the interior nodes:

$$I_{jn} = \langle \varphi_j(x), \varphi_n(x) \rangle = \frac{1}{6} \begin{cases} \Delta x_{j-1/2}, & n = j-1 \\ 2(\Delta x_{j-1/2} + \Delta x_{j+1/2}) & n = j, j+1, j, \ldots, N-1, \\ \Delta x_{j+1/2} & n = j+1 \end{cases}$$

$$J_{jn} = \langle \varphi_j(x), \frac{d \varphi_n}{d x} \rangle = \begin{cases} -1/2 & n = j-1 \\ 0 & n = j, j+1, \ldots, N-1, \\ 1/2 & n = j+1 \end{cases}.$$

For the right boundary node, only the projections with the node to its left and the self-projection between $x_{N-1} < x < x_N$ contribute, altering the inner products to:

$$I_{Nn} = \langle \varphi_N(x), \varphi_n(x) \rangle = \frac{1}{6} \begin{cases} \Delta x_{N-1/2}, & n = N-1 \\ 2\Delta x_{N-1/2}, & n = N \end{cases}$$

$$J_{Nn} = \langle \varphi_N(x), \frac{d \varphi_n}{d x} \rangle = \begin{cases} -1/2, & n = N-1 \\ 1/2, & n = N \end{cases}.$$

Defining the solution vector $q(t) = \{q_j(t), j = 1, \ldots, N\}$, the tridiagonal inner product matrices $I, J = \{I_{jn}, J_{jn}, j, n = 1, \ldots, N\}$, and the vectors $i_0, j_0 = \{I_{0n}, J_{0n}, j, n = 1, \ldots, N\}$, we can write the FEM in matrix form as

$$I_0 \frac{dq}{dt} + Jq = \begin{cases} -I_{00} \frac{dq_0}{dt} - J_{00} q_0 & j = 1 \\ 0 & j > 1. \end{cases}$$

Using trapezoidal time differencing, we obtain the desired time-discretized FEM:
\[
\frac{q^{n+1} - q^n}{\Delta t} + \frac{1}{2} J q^{n+1} + q^n = \begin{cases} 
I_0 & j = 1 \\
J_0 & j > 1 
\end{cases}
\]

The Matlab script finalp1.m on the class web page implements this tridiagonal system for time-marching the advection equation with the specified IC and BC. The first part uses the matrices \( I \) and \( J \) computed for constant grid spacing \( \Delta x = 0.01 \) (problem 1); the second part recalculates these matrices for the stretched grid \( x_j = 1 - (1 - \xi_j)^{1/2}, \xi_j = 0.02, j = 1, \ldots, 50 \).

Fig. 1 compares these two solutions at \( t = 0.9 \) with the exact solution 
\[
q(x, t) = \sin(50 \max(t - x, 0)).
\]
Both do a decent job near the left boundary, where the wave is being forced and where even the stretched grid has a resolution sufficient to resolve the wave well. The uniform-grid solution is quite respectable throughout the domain, but the stretched-grid solution starts to degrade near the right boundary where it no longer has sufficient resolution to adequately resolve a wave of wavelength \( 2\pi/50 \).

Fig. 1. FEM solutions to advection equation with uniform and stretched grids.

2. Fig. 2, generated following the instructions in the class handout, shows the \( t = 2 \) FEM solution to the wave equation IBVP generated by the Matlab PDE toolbox: