**Plane-Wave Summary**  
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A two-dimensional plane wave may be expressed as

\[
f(x, y, t) = Re \left\{ A e^{i(kx + ly - \nu t)} \right\} = Re \left\{ A e^{i\theta} \right\}
\]

(1)

- \(x, y\) and \(t\) are independent variables (space and time).
- \(k\) and \(l\) are the \(x\) and \(y\) wavenumbers (units: m\(^{-1}\)).
- \(A\) is the wave amplitude.
- \(\theta = kx + ly - \nu t\) is the wave phase angle.
- The wave propagates normal to lines of constant phase angle.

At any instant in time \([t\) fixed; \((x, y)\) varies\]

- \(\theta = kx + ly + C\); \(\theta\) is a linear function of space.
- \(\theta\) is constant on lines of \(kx + ly\).
- \(e^{i\theta} = e^{i(\theta + 2\pi n)}\), where \(n\) is an integer, are lines of equal phase (e.g. all highs, lows, nodes, etc).
- \(\vec{K} = \nabla \theta = i k + j l\) is the wave vector; \(K = |\vec{K}|\) is the wavenumber.
- \(\lambda = \frac{2\pi}{K}\) is the wavelength: the distance between neighboring lines of equal phase.
At any fixed point in space \([x,y] \) fixed; \( t \) varies

**Plot of \( \theta \) as a function of \( t \) for fixed \((x,y)\).**

- \( \theta = C - \nu t \); \( \theta \) is a linear function of time.
- \( \nu = -\frac{\partial \theta}{\partial t} \), is called the frequency: the rate that lines of constant phase pass a fixed point in space (units: \( s^{-1} \)).
- The wave period is \( \frac{2\pi}{\nu} \): length of time between points of constant phase (units: \( s \)).

**Stable and unstable waves**

If \( \theta \) has an imaginary part, \( \theta = \theta_r + i\theta_i \), then \( e^{i\theta} = e^{i(\theta_r + i\theta_i)} = e^{i\theta_r} e^{-\theta_i} \equiv A^* e^{i\theta_r} \). \( \theta_r \) is the wave phase angle as interpreted above, and \( A^* = A e^{-\theta_i} \) is a modified amplitude that depends on time and/or space. For example, if the frequency, \( \nu \), contributes the imaginary part, then the wave has time-dependent amplitude that grows or decays with time. Growing waves are called unstable, to distinguish them from the neutral waves (\( A \) = constant) that we discussed above.

**Phase speed and trace speed**

- The phase speed is the propagation speed of constant phase lines in the direction of \( \vec{K} \), \( c = \frac{\nu}{k} = -\frac{1}{|\nabla \theta|} \frac{\partial \theta}{\partial t} \) (units: \( m \ s^{-1} \)).
• The trace speeds are the speeds at which lines of constant phase propagate parallel to the coordinate axes. These speeds are never smaller than the true phase speed. The $x$ trace speed is $\nu/k$, the $y$ trace speed is $\nu/l$. Note in particular that the apparent phase speeds along each axis ($\nu/k$ and $\nu/l$) do not add vectorially to give the true phase speed ($\nu/K$).

• In principle $k$, $l$ and $\nu$ can all have arbitrary sign, but if this is allowed a redundancy is introduced in the representation of the waves. The following figure shows the four possible wave configurations that can exist for a given set of $|k|$, $|l|$, $|\nu|$. Each wave as the following characteristics:

(a) $\text{sgn}(k/l) < 0$, $\text{sgn}(\nu/k) < 0$, $\text{sgn}(\nu/l) > 0$

(b) $\text{sgn}(k/l) > 0$, $\text{sgn}(\nu/k) > 0$, $\text{sgn}(\nu/l) > 0$

(c) $\text{sgn}(k/l) > 0$, $\text{sgn}(\nu/k) < 0$, $\text{sgn}(\nu/l) < 0$

(d) $\text{sgn}(k/l) < 0$, $\text{sgn}(\nu/k) > 0$, $\text{sgn}(\nu/l) < 0$

Figure 1: Four waves with phase speed toward different quadrants and identical values of $|k|$, $|l|$, $|\nu|$.

These possibilities are conventionally covered without redundancy by stipulating that $\nu > 0$, while $k$ and $l$ may have arbitrary sign. The generalization
to three dimensions in trivial. (Other possibilities, such as demanding $k > 0$
while allowing $\nu$ and $l$ to have arbitrary sign also eliminate the redundancy
problem but are usually more awkward.)