Regular and Irregular Singular Points of ODEs

For the ODE (14.1), if $p_k(x)$, k =0, …, n–1 are not all analytic at $x = x_0$, but if $(x\text{-}x_0)^{n\text{-}k}p_k(x)$, k =0, …, n–1 are analytic at $x = x_0$, then x_0 is a *regular singular point* of the ODE ⇒ use Frobenius theory. Other singular points are called *irregular*. Even at a singular point of an ODE, some (or even all) of its solutions may be analytic, but this is not guaranteed.

Classification examples

Example 1: $y'' - \dfrac{y}{x} = 0 \quad \Rightarrow \quad p_1(x) = 0,\, p_0(x) = -\dfrac{1}{x}$

Since $p_2(x)$ is singular but $xp_0(x)$ = -1 is analytic at x_0 = 0 (and for all x), x_0 is a regular singular point.

Example 2: $y'' - \dfrac{y'}{(x+1)^2} = 0 \quad \Rightarrow \quad p_1(x) = -\dfrac{1}{(x+1)^2},\, p_0(x) = 0 \;\; (x+1)^2$

The point x_0 = -1 is an irregular singular point since $(x+1)p_1(x)$ is singular at x = -1.

Point at ∞

To classify $x_0 = \infty$, set $t = x^{-1}$ and rewrite the ODE in terms of $Y(t) = y(x)$:

$$y'(x) = \frac{dY/dt}{dx/dt} = -t^2\frac{dY}{dt}$$

$$y''(x) = -t^2\frac{d}{dt}\left(-t^2\frac{dY}{dt}\right) = t^4 Y''(t) + 2t^3 Y'(t)$$

Then Y-ODE singularity at $t_0 = 0 \leftrightarrow y$-ODE singularity at $x_0 = \infty$.

Example 3: Airy's equation

$$0 = y'' - xy \quad \Rightarrow \quad 0 = t^4 Y''(t) + 2t^3 Y'(t) - t^{-1}Y \quad \Rightarrow \quad 0 = Y''(t) + \underbrace{\frac{2}{t}}_{p_1(t)} Y' - \underbrace{\frac{1}{t^5}}_{p_0(t)} Y$$

The point t_0 = 0 is an irregular singular point since $t^2p_0(t)$ is singular at t = 0. Thus $x_0 = \infty$ is an irregular singular point of Airy's equation.

Frobenius series around regular singular points of 2nd order linear homogeneous ODEs

If a 2nd-order ODE has a regular singular point at x_0, it must have the form:

$$y'' + \frac{p(x)}{x - x_0}y' + \frac{q(x)}{\left(x - x_0\right)^2}y = 0 \qquad (15.1)$$

where $p(x)$ and $q(x)$ are analytic at x_0. Setting $p_0 = p(x_0)$, $q_0 = q(x_0)$, the ODE is well approximated as $x \to x_0$ by:

$$y'' + \frac{p_0}{x - x_0} y' + \frac{q_0}{\left(x - x_0\right)^2} y \approx 0 \qquad (15.2)$$

You may recognize this as an *equidimensional* or *Euler* ODE with solutions $(x - x_0)^\alpha$, where substitution of this form into (15.2) implies α satisfies the *indicial equation*:

$$P(\alpha) = \alpha(\alpha - 1) + p_0\alpha + q_0 = 0$$

If the roots are distinct, this gives both linearly independent solutions of (15.2); if they are equal the second linearly independent solution is $y_2(x) = (x - x_0)^\alpha \log(x - x_0)$.

A useful way to understand this somewhat strange-looking solution is as follows. Imagine that:

both roots are α_0 $\leftrightarrow$ indicial eqn is $(\alpha - \alpha_0)^2$ $\leftrightarrow$ $p_0 = -2\alpha_0+1$, $q_0 = \alpha_0^2$

Consider a slight perturbation of p_0 and/or q_0, such that the two roots α are no longer exactly equal. Then there would be two solutions, one for each root. For instance consider a perturbation $-\varepsilon^2$ to q_0. The perturbed indicial equation would be

$$0 = \alpha(\alpha - 1) + p_0\alpha + q_0 = (\alpha - \alpha_0)^2 - \varepsilon^2$$

which has roots $\alpha^\pm = \alpha_0 \pm \varepsilon$. Two linearly independent solutions are $y(x;\alpha_0 \pm \varepsilon) = (x - x_0)^{\alpha_0 \pm \varepsilon}$, regardless of how small ε is. As $\varepsilon \to 0$, both of these solutions go to $y_1(x) = (x - x_0)^{\alpha_0}$, but we can always find a second distinct linear combination with a nontrivial limit:

$$\frac{y(x;\alpha_0 + \varepsilon) - y(x;\alpha_0 - \varepsilon)}{2\varepsilon} = \frac{(x - x_0)^{\alpha_0 + \varepsilon} - (x - x_0)^{\alpha_0 - \varepsilon}}{2\varepsilon}$$

$$\underset{\varepsilon \to 0}{\to} \frac{d}{d\alpha}(x - x_0)^\alpha \bigg|_{\alpha = \alpha_0} = (x - x_0)^\alpha \log(x - x_0) = y_2(x)$$

The following theorem tells us how to use these solutions to (15.2) as the basis for convergent *Frobenius series* solutions of (15.1):

Theorem: Let $\alpha_{1,2}$ be the roots of the indicial polynomial $P(\alpha)$, ordered such that Re $\alpha_1 \geq$ Re α_2. Then the two linearly independent solutions of (15.1) are asymptotic to solutions of the leading-order equidimensional equation (15.2), and have they have series solutions that converge in $|x - x_0| < R$, where R is the distance from x_0 to the nearest other singularity of $p(x)$ or $q(x)$ in the complex plane.

Let $y(x;\alpha) = \sum_{n=0}^{\infty} a_n(\alpha)(x - x_0)^{n+\alpha}$ satisfy (15.1) at all orders in x except the leading order $O(x^{\alpha-2})$ (i. e. α is not required to satisfy the indicial equation). Then:

One series solution has the form:

$$y_1(x) = y(x;\alpha_1) = (x - x_0)^{\alpha_1} \sum_{n=0}^{\infty} a_n(\alpha_1)(x - x_0)^n$$

The form of the second series solution depends on $\alpha_1 - \alpha_2$:

(1) If $\alpha_1 - \alpha_2$ is not equal to an integer:

$$y_2(x) = y(x;\alpha_2) = (x - x_0)^{\alpha_2} \sum_{n=0}^{\infty} a_n(\alpha_2)(x - x_0)^n$$

(2) If $\alpha_1 = \alpha_2$, consider the perturbed problem with $q(x;\varepsilon) = q(x) - \varepsilon^2$, which has the perturbed indicial equation $P(\alpha;\varepsilon) = P(\alpha) - \varepsilon^2 = (\alpha - \alpha_1)^2 - \varepsilon^2$ with roots $\alpha_1 \pm \varepsilon$. Then

$$y_2(x) = \lim_{\varepsilon \to 0} \frac{y(x;\alpha_1 + \varepsilon) - y(x;\alpha_1 - \varepsilon)}{2\varepsilon} = \frac{\partial y}{\partial \alpha}(x;\alpha_1) = y_1(x)\log(x - x_0) + \sum_{n=0}^{\infty} b_n (x - x_0)^{n+\alpha_1}$$

where $b_n = da_n/d\alpha\big|_{\alpha=\alpha_1}$.

(3) If $\alpha_1 = \alpha_2 + N$, where N is a positive integer, the second series solution has the form

$$y_2(x) = (x - x_0)^{\alpha_2} \sum_{n=0}^{\infty} c_n (x - x_0)^n + \frac{\partial y}{\partial \alpha}(x;\alpha_1)$$

where the c_n are found by matching powers of x (see Bender & Orszag).

A nice feature of these Frobenius series is that we know their correct structure and convergence properties *a priori*. They provide commonly used series expansions for many familiar solutions of commonly arising singular second-order ODEs, such as Bessel functions – see example below.

Linear homogeneous ODEs of other orders with regular singular points

The same general approach works for linear homogeneous ODEs of other orders with regular singular points. For instance, consider a first order ODE of the form

$$y' + \frac{q(x)}{x - x_0} y = 0 \tag{15.3}$$

We let $q_0 = q(x_0)$ and approximate the ODE by a leading order equidimensional equation:

$$y' + \frac{q_0}{x - x_0} y \approx 0 \tag{15.4}$$

There are solutions of (15.4) in the form $(x - x_0)^\alpha$ if α satisfies the indicial equation

$$P(\alpha) = \alpha + q_0 = 0$$

There will be a Frobenius series solution to (15.3) in the form

$$y_1(x) = y(x;-q_0) \quad \text{where} \quad y(x;\alpha) = \sum_{n=0}^{\infty} a_m(\alpha)(x - x_0)^{n+\alpha}$$

As with the second-order case, this series has a radius of convergence equal to the distance from x_0 to the nearest singularity of $q(x)$ in the complex plane.

For ODEs of higher order M, there will be an M'th order indicial equation and Frobenius series solutions of the same form. If pairs of roots are equal or differ by an integer, similar strategies to the second-order case are required.

<u>Example: Bessel's equation of order ν</u>

Bessel's equation

$$y'' + \frac{y'}{x} + (1 - \frac{\nu^2}{x^2})y = 0 \qquad \text{(BE)}$$

arises in applications such as the wave equation in polar/cylindrical coordinates:

$$\frac{1}{c^2}\psi_{tt} = \nabla^2\psi = \frac{1}{r}\frac{d}{dr}r\psi_r + \frac{1}{r^2}\psi_{\theta\theta}$$

Separable solutions can be found in the form $\psi(r,\theta,t) = Y(r)e^{in\theta - i\omega t} \Rightarrow$

$$0 = \frac{1}{r}\frac{d}{dr}r\frac{dY}{dr} + \left(k^2 - \frac{n^2}{r^2}\right)Y, \quad k = \frac{\omega}{c}$$

Setting $x = kr$ and $y(x) = Y(r)$ gives Bessel's equation of order n.

Bessel's equation has a regular singular point at $x_0 = 0$, with $p(x)=1$ and $q(x) = x^2 - \nu^2$. Since these functions are entire, the Frobenius series will both converge for all x.

The indicial equation is

$$P(\alpha) = \alpha(\alpha - 1) + \underbrace{p_0}_{1}\alpha + \underbrace{q_0}_{-\nu^2} = \alpha^2 - \nu^2 \quad \Rightarrow \quad \alpha_1 = \nu, \alpha_2 = -\nu$$